Massachusetts Institute of Technology

Subject 6.651J/8.613J/22.611J — R. Parker
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Problem Set 9

Problem 1.

In this problem we'll develop a general (if a bit formal) derivation of the Vlasov equation.

As we discussed in class, conservation of particles requires

$$\frac{D(f\Delta\vec{r}\Delta\vec{v})}{Dt} = 0,$$

where the symbol *D*/*Dt* means that the time derivative is to be taken along a particle orbit in phase space. Specifically,

$$\frac{D(f)}{Dt} = \lim(\Delta t \to 0)\,\frac{f(\vec{r}+\vec{v}\Delta t, \vec{v}+\vec{a}(\vec{r},\vec{v},t)\Delta t, t+\Delta t) - f(\vec{r},\vec{v},t)}{\Delta t}.$$

a) Show that the RHS of this expression reduces to

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\nabla f + \vec{a}\cdot\nabla_v f$$

b) Now consider the term

$$\frac{D(\Delta\vec{r}\Delta\vec{v})}{Dt} = \lim(\Delta t \to 0)\,\frac{\Delta\vec{r}\Delta\vec{v}_{\vec{r}+\vec{v}\Delta t,\vec{v}+\vec{a}\Delta t,t+\Delta t} - \Delta\vec{r}\Delta\vec{v}_{\vec{r},\vec{v},t}}{\Delta t}$$

As indicated, the rate of change of the phase space volume is to be calculated along a particle trajectory in phase space. For small Δt along this trajectory, the particle orbit is simply given by

$$\vec{r}\,' = \vec{r} + \vec{v}\Delta t$$
$$\vec{v}\,' = \vec{v} + \vec{a}(\vec{r},\vec{v},t)\Delta t$$

These equations define a simple transformation of variables between the $\vec{r},\vec{v}$ and $\vec{r}\,',\vec{v}\,'$ coordinates. Accordingly, a well-known mathematical result is that the volume elements are related by the Jacobian of the transformation defined as the determinant of the matrix

$$\left[c_{ij} = \frac{\partial x'_j}{\partial x_i}\right].$$

Use this result to calculate $\frac{D(\Delta\vec{r}\Delta\vec{v})}{Dt}$ and complete the derivation of Vlasov's equation.

Problem 2.

In deriving the fluid energy equation in class, we got to the following point:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}mnu^2+\frac{3}{2}nkT\right)+\nabla\cdot\left(\frac{1}{2}mnu^2\vec{u}+\frac{5}{2}nkT\vec{u}+\vec{\vec{\Pi}}\cdot\vec{u}\right)+\nabla\cdot\vec{q}-q\vec{E}\cdot n\vec{u}=\frac{m}{2}\int d\vec{v}v^2\sum_\beta C(f,f_\beta)\,.$$

It was then stated that this equation could be transformed into the simpler form:

$$\frac{3}{2}n\frac{dkT}{dt}+p\nabla\cdot\vec{u}=-\Pi_{ij}\frac{\partial u_i}{\partial x_j}-\nabla\cdot\vec{q}+Q\,,$$

where the summation convention applies and

$$Q=\frac{m}{2}\int d\vec{w}w^2\sum_\beta C(f,f_\beta)\,.$$

Show that the second form follows from the first by using the continuity equation, the result of dotting the momentum equation with $\vec{u}$, and the identity

$$-\vec{u}\cdot(\nabla\cdot\vec{\vec{\Pi}})+\nabla\cdot(\vec{\vec{\Pi}}\cdot\vec{u})=\Pi_{ij}\frac{\partial u_i}{\partial x_j}\,.$$

Problem 3.

Consider 1-D plasma oscillations proportional to $\exp(-i\omega t+ikx)$ in a hot plasma with a 1-D electron distribution function given by

$$\tilde{f}_e(v_x)=\frac{v_e}{\pi}\frac{1}{v_x^2+v_e^2}\,.$$

For simplicity assume that k is real, but that ω could be complex.

a) Determine an algebraic dispersion relation for electron oscillations, assuming that the ions are immobile.

b) Solve the dispersion relation obtained in a) for $\omega(k)$.

c) Now assume that the ions have a distribution function given by

$$\tilde{f}_i(v_x)=\frac{v_i}{\pi}\frac{1}{v_x^2+v_i^2}\,,$$

while the electron distribution function is the same as in part a). Assuming $\omega / k << v_e$, determine $\omega(k)$ for ion acoustic waves.

Possibly useful integrals:

$$\int_{-\infty}^{\infty} \frac{v_x}{(v_x^2+v_e^2)^2} \frac{1}{v_x-\varsigma} dv_x = -\frac{\pi}{2v_e} \left\{ \frac{(\varsigma - i v_e)^2}{(\varsigma^2+v_e^2)^2} \right\} \quad \mathrm{Im}\,\varsigma > 0$$

$$\int_{-\infty}^{\infty} \frac{v_x}{(v_x^2+v_e^2)^2} \frac{1}{v_x-\varsigma} dv_x = -\frac{\pi}{2v_e} \left\{ \frac{(\varsigma + i v_e)^2}{(\varsigma^2+v_e^2)^2} \right\} \quad \mathrm{Im}\,\varsigma < 0$$

$$\int_{-\infty}^{\infty} \frac{v_x}{(v_x^2+v_e^2)} \frac{1}{v_x-\varsigma} dv_x = \pi \left\{ \frac{(v_e + i\varsigma)}{(v_e^2+\varsigma^2)} \right\} \quad \mathrm{Im}\,\varsigma > 0$$

$$\int_{-\infty}^{\infty} \frac{v_x}{(v_x^2+v_e^2)} \frac{1}{v_x-\varsigma} dv_x = \pi \left\{ \frac{(v_e - i\varsigma)}{(v_e^2+\varsigma^2)} \right\} \quad \mathrm{Im}\,\varsigma < 0$$

$$\int_{-\infty}^{\infty} \frac{v_x^2}{(v_x^2+v_e^2)^2} \frac{1}{v_x-\varsigma} dv_x = -\frac{\pi}{2v_e} \varsigma \left\{ \frac{(\varsigma - i v_e)^2}{(\varsigma^2+v_e^2)^2} \right\} \quad \mathrm{Im}\,\varsigma > 0$$

$$\int_{-\infty}^{\infty} \frac{v_x^2}{(v_x^2+v_e^2)^2} \frac{1}{v_x-\varsigma} dv_x = -\frac{\pi}{2v_e} \varsigma \left\{ \frac{(\varsigma + i v_e)^2}{(\varsigma^2+v_e^2)^2} \right\} \quad \mathrm{Im}\,\varsigma < 0$$