8.012 Physics I: Classical Mechanics
Fall 2008

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

Physics 8.012 Fall 2006

Final Exam
Monday, December 18, 2006

NAME: S O L U T I O N S

MIT ID number: ________________________________

Instructions:

1. Do all **EIGHT (8)** problems. You have **3 hours**.
2. **SHOW ALL WORK**. Be sure to **circle your final answer**.
3. Read the questions carefully.
4. All work must be done in the blue/white books provided.
5. NO books, notes, calculators or computers are permitted. **A list of useful equations is provided on the last page**.

Your Scores

Problem	Maximum	Score	Grader
1	10		
2	15		
3	10		
4	10		
5	15		
6	15		
7	15		
8	10		
Total	100		

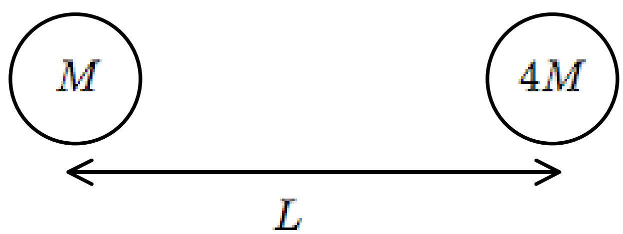
Problem 1: Short Answer Problems [2 pts each]

(a) List Newton's three laws (order is not important).

(b) What are the units of impulse, using the notation [L] = length, [M] = mass and [T] = time?

(c) An archer in the northern hemisphere on Earth faces North and shoots an arrow at very high velocity. Will the arrow veer off to the left or the right? Justify your answer (note that the angular velocity vector of the Earth points away from the North Pole).

(d) Referring to the figure to the right, how far from the mass on the left is the point at which a test particle feels equal gravitational force from both spheres? Assume L is much larger than the radii of the spheres



(e) A mass is at rest in an inertial frame at a distance R from the origin. An observer sits at the origin of a rotating frame with angular velocity Ω that is initially aligned with the inertial frame. Describe in words the motion of the object from the observer's perspective, and explain how that motion comes about in terms of fictional forces (there are no real forces acting on the mass).

(a) (i) An object at rest or in uniform motion tends to stay at rest or in uniform motion unless acted on by an outside force (e.g., inertial frames exist)

(ii) $\vec{F} = m\vec{a}$ - acceleration is proportional to force and inversely proportional to mass

(iii) $\vec{F}_{12} = -\vec{F}_{21}$ - force exerted by one body onto a second body is equal and directed in the opposite direction to the force exerted by the second body onto the first

(b) Impulse = FxΔt = Δp = $[M]\,[L]\,[T]^{-1}$

(c) The veering of the arrow results from Coriolis force and is directed toward the **right**. $-2m(\vec{\Omega} \times \vec{v}) \propto -(\hat{z} \times \hat{north}) \propto \hat{east}$ which is to the right when facing north.

(d) Total forces cancel at a distance x from the left sphere when:

$$-\frac{GM}{x^2} + \frac{4GM}{(L-x)^2} = 0$$

$$\Rightarrow (L-x)^2 = 4x^2$$

$$\Rightarrow 3x^2 + 2xL - L^2 = 0$$

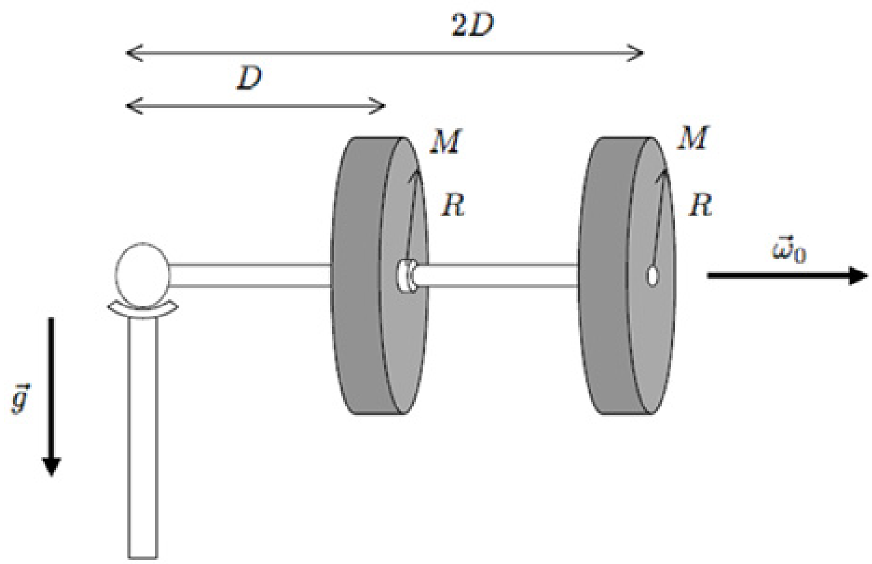
This quadratic equation is solved for **x = L/3** (the negative solution, x=-L, does not correspond to a point where there is zero net force).

(e) To the observer in the rotating frame, the mass would appear to move in a circle of radius R about origin with velocity v = RΩ in the opposite sense of the rotation of the frame. This is consistent with the fictional forces experienced by the mass as both centrifugal and coriolis accelerations contribute, giving

$$\vec{a} = \Omega^2 R\hat{r} - 2\Omega v\hat{r} = -\Omega^2 R\hat{r}$$

This net (fictional) force provides the necessary *centripetal* acceleration to keep the object moving in a fixed radius circle.

Problem 2: The Double Gyroscope [15 pts]



A gyroscope consists of two identical uniform disks with mass M and radius R mounted on a rigid axle with length 2D. The axle is fixed to the outer disk, while the inner disk is allowed to spin freely but is constrained to remain at a distance D from the pivot (at left) by a collar. The axle spins about the pivot freely on a frictionless mount. The outer disk and axle are initially spun up to an angular frequency ω_0. Assume that the mass of the axle and pivot mount are negligible. Gravity points downward. Ignore nutation.

(a) Calculate the precession rate Ω of the gyroscope assuming that the inner disk is not spinning. Assume that ω_0 is very large and the gyroscopic approximation applies.

(b) Friction between the axle and the disk closest to the pivot causes the latter to spin up. Assuming that a constant torque τ acts at the interface between the axle and inner hole of this disk, calculate the final spin rate ω of both disks and the final precession rate in terms of the initial precession rate.

(c) How much energy is lost from the gyroscope system during the spinning up of the inner disk?

(a) The net torque on the gyroscope with respect to the pivot point is

$$\vec{\tau} = \vec{r} \times \vec{F} = (MgD + 2MgD)\hat{\otimes} = 3MgD\hat{\otimes}$$

The gyroscope approximation is:

$$|\vec{\tau}| = \left|\frac{d\vec{L}}{dt}\right| = |\vec{L}|\,\Omega$$

The angular momentum of the first disk is

$$|\vec{L}| = \frac{1}{2}MR^2\omega_0$$

Hence

$$3MgD = \frac{1}{2}MR^2\omega_0\Omega$$

$$\Rightarrow \boxed{\Omega = \frac{6gD}{R^2\omega_0}}$$

(b) The constant frictional torque is

$$\tau_f = \frac{dL}{dt} = I\frac{d\omega}{dt} \Rightarrow \frac{d\omega}{dt} = constant$$

Hence:

$$\omega_{in} = \frac{\tau_f}{I}t$$

This is the result of the force from the axle acting on the inner wheel; Newton's third law tells us that an equal and opposite force acts on the axle + outer wheel and slows it down. Hence

$$\omega_{out} = \omega_0 - \frac{\tau_f}{I}t$$

The frictional force ceases when the surfaces do not slide relative to one another, i.e., when $\omega_{in} = \omega_{out} = \omega$, so

$$\boxed{\omega = \frac{1}{2}\omega_0}$$

To calculate the precession, note that the torque hasn't changed, while the magnitude of the angular momentum vector is

$$|\vec{L}| = 2 \times \frac{1}{2}MR^2\omega = \frac{1}{2}MR^2\omega_0$$

and is therefore also unchanged. So the precession rate is the same:

$$\boxed{\Omega = \frac{6gD}{R^2\omega_0}}$$

(c) Only the rotational kinetic energy of the system changes as a result of the inner wheel spinning up. The initial rotational kinetic energy is

$$E_i = \frac{1}{2}I_\perp\Omega^2 + \frac{1}{2}I\omega_0^2$$

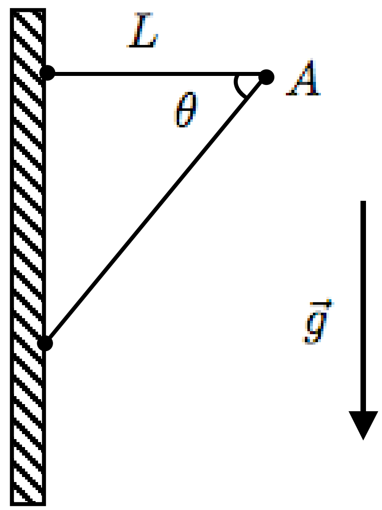
where $I_\perp$ is the moment of inertia of the entire gyroscope rotating about the pivot. The final rotational kinetic energy is

$$E_f = \frac{1}{2}I_\perp\Omega^2 + 2 \times \frac{1}{2}I\omega^2 = \frac{1}{2}I_\perp\Omega^2 + \frac{1}{4}I\omega_0^2$$

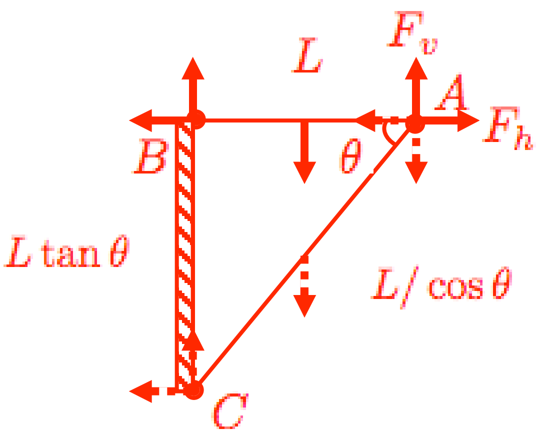
Hence the total energy lost is:

$$\Delta E = E_i - E_f = \frac{1}{4}I\omega_0^2 = \boxed{\frac{1}{8}MR^2\omega_0^2}$$

Problem 3: Balanced Poles [10 pts]



Two sticks are attached with frictionless hinges to each other and to a wall, as shown above. The angle between the sticks is θ. Both sticks have the same constant linear mass density λ, and the horizontal stick has length L. Find the force (both horizontal and vertical components) that the lower sticks applies to the upper one where they connect at point A. Assume gravity points downward.



The best way to approach this problem is to balance the forces and torques acting on the sticks, since the system is not accelerating and not rotating. A force diagram is shown on the left, indicating forces acting on the horizontal stick as solid arrows and forces acting on the angled stick as dotted arrows. The objective is to determine F_h and F_v.

First measure the torque on the horizontal stick about point B (assuming torque is positive in a clockwise direction on the page):

$$\tau_B = \lambda L g\frac{L}{2} - F_v L = 0 \Rightarrow \boxed{F_v = \frac{1}{2}\lambda L g}$$

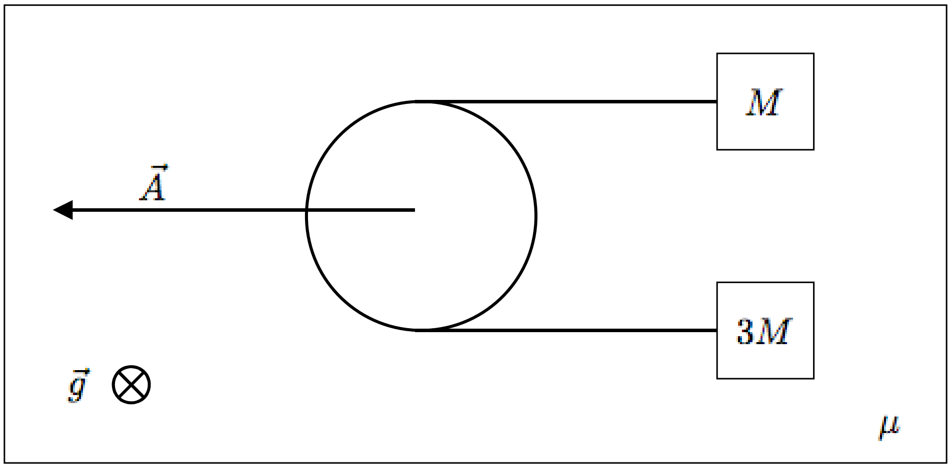
Now measure the torque on the angled stick about point B:

$$\tau_C = \lambda\frac{L}{\cos\theta}g\frac{L}{2} + F_v L - F_h L\tan\theta = 0$$

$$\Rightarrow F_h\tan\theta = \frac{\lambda L g}{2\cos\theta} + \frac{1}{2}\lambda L g$$

$$\Rightarrow \boxed{F_h = \frac{\lambda L g}{2}\left(\frac{1}{\sin\theta} + \frac{1}{\tan\theta}\right)}$$

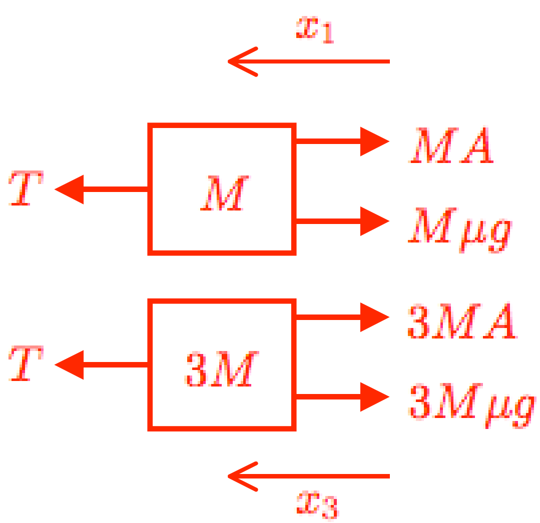
Problem 4: The Accelerated Atwood Machine [10 pts]



An idealized Atwood machine, consisting of a two blocks of masses M and 3M connected via a massless string through a massless pulley, sits on a flat horizontal table. The coefficient of kinetic friction between the block and table surfaces is μ. The pulley is pulled by a string attached to its center and accelerated to the left. Assume that gravity acts with constant acceleration g down through the plane of the table.

(a) What are the horizontal accelerations of the two masses in the frame of rest of the table?

(b) What is the maximum acceleration A for which the block of mass 3M will remain stationary?



This problem is straightforward if one works in the accelerated reference frame. The force diagrams for the two masses are shown at left. Note the additional fictional forces, and the fact that the friction forces both act toward the right – this is because in the accelerated frame the table is sliding to the right, dragging the blocks with it. The tension force acts via the massless string and is the same for both masses.

The equations of motion for the two blocks are:

$$M\ddot{x}_1 = T - MA - \mu Mg$$
$$M\ddot{x}_3 = T - 3MA - 3\mu Mg$$

The positions of the masses are constratined to $x_1 + x_3$ = constant due to the fixed length of rope connecting them; hence

$$\ddot{x}_1 + \ddot{x}_3 = 0$$
$$\Rightarrow \frac{T}{M} - A - \mu g + \frac{T}{3M} - A - \mu g = 0$$
$$\Rightarrow \frac{T}{M} = \frac{3}{2}A + \frac{3}{2}\mu g$$

Replacing this expression in to the equations of motion yield:

$$\ddot{x}_1 = \frac{1}{2}(A + \mu g) = -\ddot{x}_3$$

We move to the frame of rest of the table by adding the acceleration A to both equations of motion:

$$\ddot{x}_1' = \frac{3}{2}A + \frac{1}{2}\mu g$$
$$\ddot{x}_3' = \frac{1}{2}A - \frac{1}{2}\mu g$$

(b) For the 3M mass to remain stationary in the frame of rest, $\ddot{x}_3' = 0$ so

$$A = \mu g$$

<u>Problem 5: Central Force Potentials</u> [15 pts]

A particle of mass m is inserted into a central force field of the form

$$\vec{F} = -\frac{k}{r^n}\hat{r}$$

where r is the radial distance from the origin, and k and n are constants.

(a) Show that the total angular momentum $\vec{L}$ about the origin of the system is a constant of the motion.

(b) Derive the general effective potential for the system, assuming $|\vec{L}| = L$.

(c) Derive a general expression for the equilibrium point(s) for this potential.

(d) For what values of n are stable orbits possible; i.e., for which the particle is constrained to a finite range of r? Show your results in detail (i.e,. do not simply state the answer).

(a) The torque on the mass about the origin is:

$$\vec{\tau} = \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = (-\frac{k}{r^{n-1}})(\hat{r} \times \hat{r}) = 0$$

Hence, the angular momentum vector about the origin is a constant of the motion.

(b) The effective potential is

$$U_{eff}(r) = U(r) + \frac{L^2}{2mr^2}$$

where

$$U(r) = -\int \vec{F}(r) \cdot d\vec{r} = \int \frac{k}{r^n}(\hat{r} \cdot \hat{r})dr = -\frac{1}{n-1}\frac{k}{r^{n-1}} + C$$

We can choose a zeropoint such that U(∞) = 0, which for n > 1 implies C = 0, and hence:

$$\boxed{U_{eff}(r) = -\frac{1}{n-1}\frac{k}{r^{n-1}} + \frac{L^2}{2mr^2}}$$

(c) The equilibrium point satisfies

$$\frac{dU_{eff}}{dr}|_{r_{eq}} = 0$$

$$\Rightarrow \frac{k}{r_{eq}^n} - \frac{L^2}{mr_{eq}^2} = 0$$

$$\Rightarrow r_{eq}^{n-3} = \frac{mk}{L^2}$$

$$\Rightarrow \boxed{r_{eq} = \left(\frac{mk}{L^2}\right)^{\frac{1}{n-3}}}$$

(d) For a stable orbit, the equilibrium point should be a stable one which requires

$$\frac{d^2U_{eff}}{dr^2}|_{r_{eq}} > 0$$

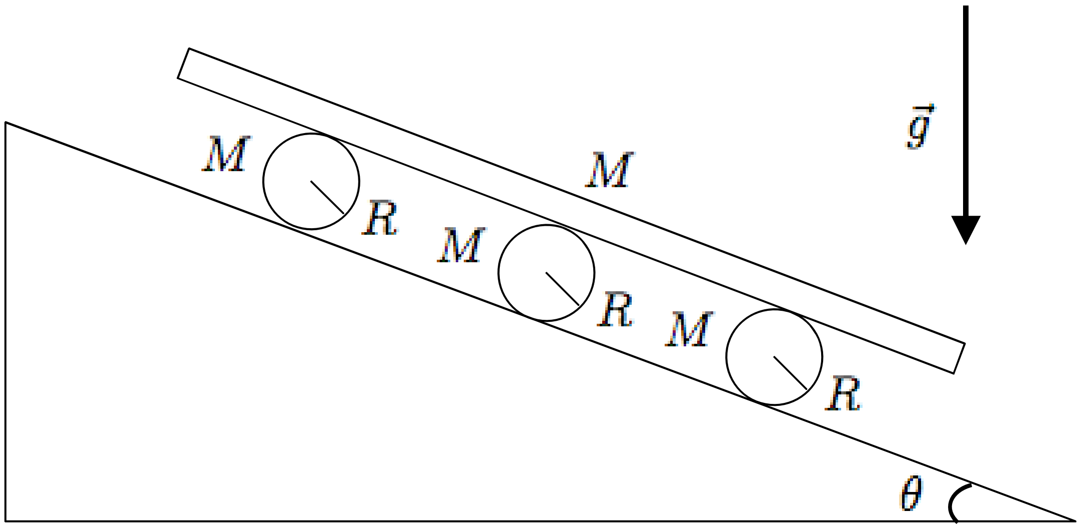
$$\Rightarrow -\frac{nk}{r_{eq}^{n+1}} + \frac{3L^2}{2mr_{eq}^4} > 0$$

$$\Rightarrow r_{eq}^{n-3} > \frac{nkm}{3L^2}$$

$$\Rightarrow \frac{mk}{L^2} > \frac{nkm}{3L^2}$$

$$\Rightarrow \boxed{n < 3}$$

Problem 6: Rolling Platform [15 pts]



A platform of mass M and uniform density rests on three solid cylinders, each of mass M, radius R and uniform density. The whole structure is initially at rest on an inclined plane tilted at angle θ, and then released. Assume that there is no slipping between the various surfaces and gravity points downward.

(a) What is the acceleration of the platform along the incline?

(b) If the solid cylinders are replaced with hollow cylinders, what is the resulting acceleration of the platform?

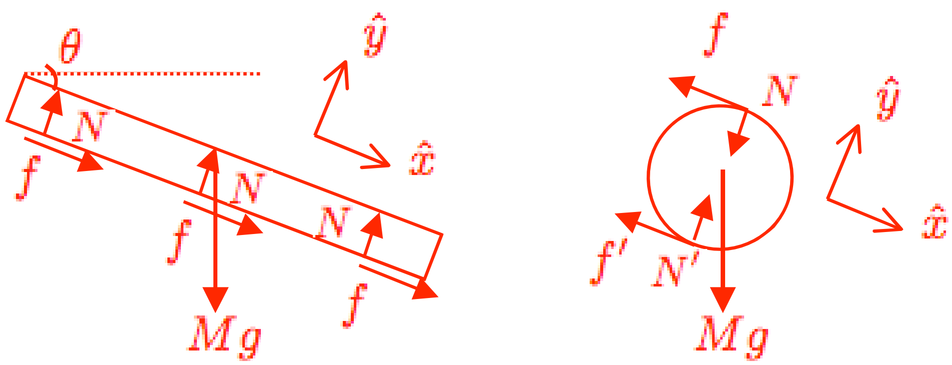
(c) Based on the result of part (b), what are the optimal rollers for moving a heavy platform such as the one shown above: (1) solid uniform cylinders, (2) hollow cylinders, or (3) cylinders with mass concentrated at their centers? Justify your answer (This would have been an engineering consideration for the ancient Egyptians when they transported large blocks of stone to construct the Pyramids.)

(a) There are two main points to this problem that must be recognized to solve it. The first is that friction must act between the wheels and platform, and between the wheels and incline, and that the former modifies the acceleration of the platform. The second is that the no slipping constraint applies between the wheels and platform and the wheels and incline, so

$$\ddot{x}_p = \ddot{x}_w + R\ddot{\phi} = 2R\ddot{\phi} = 2\ddot{x}_w$$

Where ϕ is the angular motion of the wheel.

This problem can be solved in two ways: by solving the translational and rotational equations of motion and applying the constraint equations and using energy.



The first solution is built upon the force diagrams shown above. Note the matching of forces between the wheel and platform (*f* and N). Also note that we have not assumed *f* = *f*', nor have we assumed a relation between *f* and N based on the (unknown) coefficient of friction. We can restrict analysis to the platform and one wheel, since the other wheels will experience the same motion. The translational equations of motion for the platform are (note the directions of x and y from the figures):

$$M\ddot{x}_p = 3f + Mg\sin\theta$$

$$M\ddot{y}_p = 3N - Mg\cos\theta = 0$$

The translational equations of motion for each wheel are

$$M\ddot{x}_w = -f - f' + Mg\sin\theta$$

$$M\ddot{y}_w = N' - N - Mg\cos\theta = 0$$

The wheel also spins, so we need to calculate rotational equations of motion. To do this we choose a fixed point parallel to the incline that is aligned with center of the wheel – we cannot chose the center of the wheel as our reference point because the wheel is accelerating down the incline. Hence,

$$I\ddot{\phi} = Rf' - Rf + x_w N - x_w N' + x_w Mg\cos\theta = Rf' - Rf$$

where the x_w terms have dropped out based on the y-axis equation of motion of the wheel above.

Now we apply the constraint equations to the x_p, x_w and ϕ equations of motion, and assume $I = MR^2/2$ giving:

$$\ddot{x}_p = -\frac{3}{M}f + g\sin\theta$$

$$2\ddot{x}_w = \ddot{x}_p = -\frac{2}{M}(f + f') + 2g\sin\theta$$

$$2R\ddot{\phi} = \ddot{x}_p = \frac{4}{M}(f' - f)$$

Using these equations to eliminate *f* and *f'* yields:

$$\boxed{\ddot{x}_p = \frac{20}{17}g\sin\theta}$$

The alternate derivation of the solution involves computing the change in energy for the system, which is conserved since the friction forces do no work (the surfaces are not slipping relative to each other). Assuming that the system starts from rest at $x_p = x_w = 0$, then

$$\Delta E = \Delta U + \Delta K$$
$$= 3Mg\Delta h_w + Mg\Delta h_p + 3 \times \frac{1}{2}M\dot{x}_w^2 + 3 \times \frac{1}{2}I\dot{\phi}^2 + \frac{1}{2}M\dot{x}_p^2$$
$$= -3Mgx_w \sin\theta - Mgx_p \sin\theta + \frac{3}{8}M\dot{x}_p^2 + \frac{3}{16}M\dot{x}_p^2 + \frac{1}{2}M\dot{x}_p^2 = 0$$

where the constraint equations have been used in the last step. Taking a time derivative of this equation, and again using the constraint equations gives:

$$3Mg\dot{x}_w \sin\theta + Mg\dot{x}_p \sin\theta = \tfrac{5}{2}Mg\dot{x}_p \sin\theta = \tfrac{17}{8}M\dot{x}_p\ddot{x}_p$$
$$\Rightarrow \ddot{x}_p = \frac{20}{17}g\sin\theta$$

(b) Changing the moment of inertia to $I = MR^2$ changes the rotational equation of motion in the first solution to:

$$2R\ddot{\theta} = \ddot{x}_p = \frac{2}{M}(f' - f)$$

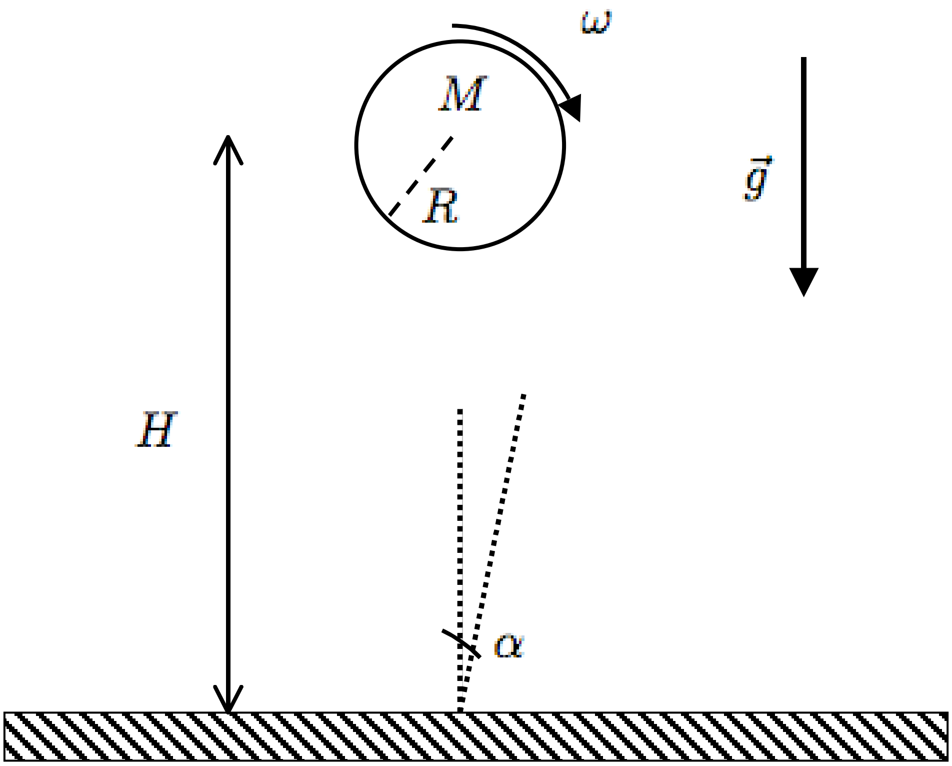
This results in $f = 0$, so

$$\ddot{x}_p = g\sin\theta$$

Similarly, we could replace the rotational energy term in the second solution and derive the same result.

(c) This question is subjective, but one may consider that an optimal engineering choice is to obtain the most acceleration for the least force applied. Since the force applied must also compensate for the motion of the rollers, the minimum moment of inertia for the rollers is optimal, which would be the case for those with mass concentrated at the center.

Problem 7: Spinning Bouncing Ball [15 pts]



A uniform sphere of mass M and radius R spinning with angular velocity ω is dropped from a height H. It bounces on the floor and recoils with the same vertical velocity. During the bounce, the surface of the ball slips relative to the surface of the floor (i.e., it does not roll) and in the process the ball is acted upon by a friction force with magnitude $f = \mu N$, where N is the normal contact force between the ball and the floor and μ is a constant. Hence, the ball experiences impulses in both vertical and horizontal directions. Assume that the duration of contact, Δt, is very short.

(a) What angle α with respect to vertical does the ball recoil?

(b) What is the final rotation velocity of the ball?

(c) What value of H results in the ball bouncing off with no spin?

(a) The angle α is related to the horizontal and vertical components of the ball's velocity after the bounce. The vertical component is straightforward, as it is stated in the problem that it rebounds with the same velocity as it struck the ground, which is:

$$v_y = \sqrt{2gH}$$

The horizontal component arise from the impulse due to friction acting at the contact surface:

$$v_x = \frac{\Delta p_x}{m} \approx \frac{\bar{F}_x \Delta t}{m} = \frac{\mu \bar{N} \Delta t}{m}$$

This is related to the vertical impulse by:

$$\Delta p_y = \bar{N}\Delta t = 2mv_y = 2m\sqrt{2gH}$$

Hence:

$$v_x = 2\mu\sqrt{2gH}$$

and:

$$\boxed{\tan\alpha = \frac{v_x}{v_y} = 2\mu}$$

(b) The frictional force produces a torque that acts in the opposite direction as the spinning of the ball. This torque exerts a rotational impulse (change in angular momentum):

$$\Delta L = I\Delta\omega \approx -\bar{\tau}\Delta t = -R\mu\bar{N}\Delta t = -2R\mu m\sqrt{2gH}$$

Hence,

$$\omega - \omega_0 = \Delta\omega = -\frac{2R\mu m\sqrt{2gH}}{I}$$

Using the momentum of inertia for a sphere:

$$I = \frac{2}{5}MR^2$$

yields

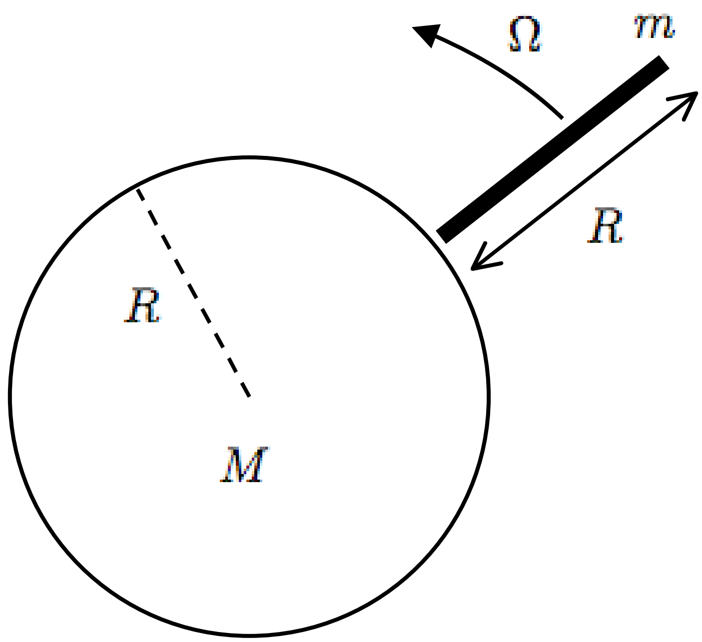
$$\boxed{\omega = \omega_0 - \frac{5\mu\sqrt{2gH}}{R}}$$

(c) For no spin after the bounce, ω = 0, so

$$\omega_0 = \frac{5\mu\sqrt{2gH}}{R}$$

$$\Rightarrow \boxed{H = \frac{R^2\omega_0^2}{50\mu^2 g}}$$

Problem 8: Orbiting Rope [10 pts]



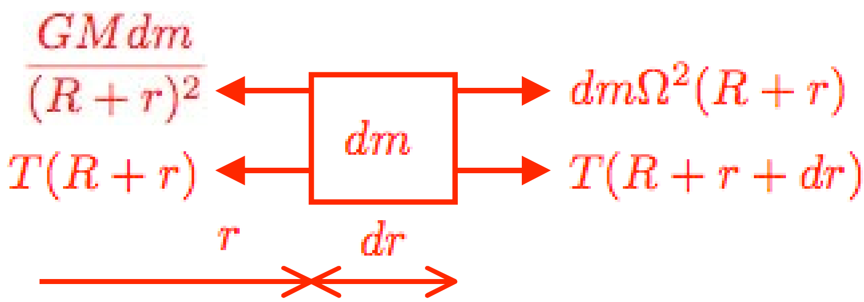
A uniform rope of mass m and length R orbits a spherical planet of mass M and radius R with constant angular velocity Ω, such that the rope remains straight and in a circular orbit by a balance of gravitational and centrifugal forces. For this problem, ignore the effects of air resistance and assume that M >> m.

(a) Derive an expression for the angular velocity Ω as a function of m, M and R (not all three quantities may be in the expression). Hint: consider the tension in the rope as a function of radius.

(b) What point on the rope experiences the greatest tension (and is thus most likely to break)?

(c) An orbiting rope such as this is one concept for a space elevator, allowing material to be transported into orbit without the use of rockets. What restrictions must be made on the angular velocity Ω and the anchor point of the rope on the Earth in this case?

(a) The best way to approach this problem is to take the hint and consider how tension varies along the rope. Consider a small section of rope of length dr and mass dm at radius r, and assume a coordinate system that rotates with the rope. The forces acting on that piece of rope, including gravitational, centrifugal (fictional) and radius-dependent tension, are illustrated below. Note that Coriolis force does not act because the rope has no velocity in the rotating frame of reference.



The net force acting on this piece of rope must be equal to zero:

$$T(R+r+dr) - T(R+r) + dm\Omega^2(R+r) - \frac{GMdm}{(R+r)^2} = 0$$

$$\Rightarrow dT = dm\left[\frac{GM}{(R+r)^2} - \Omega^2(R+r)\right]$$

where dT is the differential change in the tension along the rope. Taking the intergral of this expression:

$$\int_0^{T(r)} dT = T(r) = \int_0^r \frac{dm}{dr}dr\left[\frac{GM}{(R+r)^2} - \Omega^2(R+r)\right]$$

$$= -\lambda\left[\frac{GM}{(R+r)} + \frac{1}{2}\Omega^2(R+r)^2\right]_0^r$$

The tension at the ends of the rope must vanish. We have explicitly assumed that the T(0) = 0, so now consider that the tension at the far end T(R) = 0; then

$$\frac{GM}{R}\left(\frac{1}{2}-1\right)+\frac{1}{2}\Omega^2(4R^2-R^2)=0$$

$$\Rightarrow \boxed{\Omega^2=\frac{GM}{3R^3}}$$

(b) Extremum point in tension occurs when dT/dr = 0, or

$$\frac{dT}{dr}=\frac{dm}{dr}\left[\frac{GM}{(R+r)^2}-\Omega^2(R+r)\right]=0$$

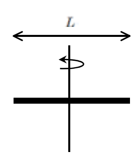
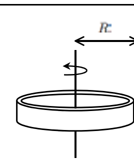
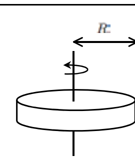
$$\Rightarrow \frac{GM}{(R+r)^2}=\frac{GM}{3R^3}(R+r)$$

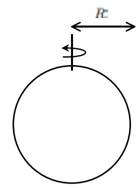
$$\Rightarrow (R+r)^3=3R^3$$

$$\Rightarrow \boxed{r=R(3^{\frac{1}{3}}-1)\approx 0.44R}$$

(c) For this kind of space elevator to work, the end point closest to the ground must stay at the same spot relative to the Earth on its own, otherwise it will be too hard to attach anything to the rope, and if the rope were anchored to the ground it would quickly get wrapped up and would likely fall to the Earth. Hence, the angular velocity should match the angular rotation rate of the Earth. Also, the anchor spot should be as close to the equator as possible, for otherwise the end of the rope will drift north to south as it completes a full orbit (this is why gyrosynchronous satellites tend to be above the equator). Note that many effects limit the feasibility of a space elevator that we've ignored here, such air resistance and winds, nutation effects on spin axis of the Earth, deviations of the Earth's shape from a perfect sphere and the tensile strength of the rope.

USEFUL EQUATIONS

Trajectory for constant acceleration	$\vec{r}(t) = \frac{1}{2}\vec{a}t^2 + \vec{v}(0)t + \vec{r}(0)$
Velocity in polar coordinates	$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$
Acceleration in polar coordinates	$\ddot{\vec{r}} = \ddot{r}\hat{r} + r\ddot{\theta}\hat{\theta} + 2\dot{r}\dot{\theta}\hat{\theta} - r\dot{\theta}^2\hat{r}$
Center of mass of a rigid body	$\vec{R} = \frac{\sum m_i\vec{r}_i}{\sum m_i} = \frac{1}{M}\int \rho\vec{r}dV$
Kinetic energy	$K = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$
Work	$W = -\int \vec{F}\cdot d\vec{r} = -\int \mid\vec{\tau}\mid d\theta$
Angular momentum	$\vec{L} = \vec{r}\times\vec{p} = I\dot{\theta}$
Torque	$\vec{\tau} = \vec{r}\times\vec{F} = I\ddot{\theta}$
Moment of inertia for a uniform bar	 $I = \frac{1}{12}ML^2$
Moment of inertia for a uniform hoop	 $I = MR^2$
Moment of inertia for a uniform disk	 $I = \frac{1}{2}MR^2$

Moment of inertia for a uniform sphere	 $I = \frac{2}{5} M R^2$
Parallel axis theorem	$I = I_{COM} + MR^2$
Velocity from rotation	$\vec{v} = \vec{\omega} \times \vec{r}$
Moments of inertia tensor (permutate x→y→z)	$I_{xx} = \sum_i m_i (y_i^2 + z_i^2)$
Products of inertia tensor (permutate x→y→z)	$I_{xy} = -\sum_i m_i x_i y_i$
Euler's equations (permutate 1→2→3)	$\tau_1 = I_1 \frac{d\omega_1}{dt} + (I_3 - I_2)\omega_2\omega_3$
Fictitious force in an accelerating frame	$\vec{F}_f = -m\vec{A}$
Fictional forces in a rotating frame	$\vec{F}_f = -2m\vec{\Omega} \times \vec{v}_{rot} - m\vec{\Omega} \times (\vec{\Omega} \times \vec{r})$
Time derivative of an arbitrary vector between inertial and rotating frames	$\frac{d\vec{B}}{dt}_{(in)} = \frac{d\vec{B}}{dt}_{(rot)} + \vec{\Omega} \times \vec{B}$
Reduced mass	$\mu = \frac{m_1 m_2}{m_1 + m_2}$
Effective potential for orbital motion	$U_{eff}(r) = U(r) + \frac{l^2}{2\mu r^2}$