13.2 Kinetic Energy

The first form of energy that we will study is an energy associated with the coherent motion of molecules that constitute a body of mass m; this energy is called the *kinetic energy* (from the Greek word *kinetikos* which translates as *moving*). Let us consider a car moving along a straight road (along which we will place the x-axis). For an observer at rest with respect to the ground, the car has velocity $\vec{\mathbf{v}} = v_x \hat{\mathbf{i}}$. The speed of the car is the magnitude of the velocity, $v \equiv |v_x|$.

> *The **kinetic energy** K of a non-rotating body of mass m moving with speed v is defined to be the positive scalar quantity*

$$K \equiv \frac{1}{2} m v^2 \tag{13.2.1}$$

The kinetic energy is proportional to the square of the speed. The SI units for kinetic energy are $[\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-2}]$. This combination of units is defined to be a joule and is denoted by $[\mathrm{J}]$, thus $1\,\mathrm{J} \equiv 1\,\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-2}$. (The SI unit of energy is named for James Prescott Joule.) The above definition of kinetic energy does not refer to any direction of motion, just the speed of the body.

Let's consider a case in which our car changes velocity. For our initial state, the car moves with an initial velocity $\vec{\mathbf{v}}_i = v_{x,i}\,\hat{\mathbf{i}}$ along the x-axis. For the final state (at some later time), the car has changed its velocity and now moves with a final velocity $\vec{\mathbf{v}}_f = v_{x,f}\,\hat{\mathbf{i}}$. Therefore the change in the kinetic energy is

$$\Delta K = \frac{1}{2} m v_f^{\,2} - \frac{1}{2} m v_i^{\,2}. \tag{13.2.2}$$

Example 13.1 Change in Kinetic Energy of a Car

Suppose car A increases its speed from 10 to 20 mph and car B increases its speed from 50 to 60 mph. Both cars have the same mass m. (a) What is the ratio of the change of kinetic energy of car B to the change of kinetic energy of car A? In particular, which car has a greater change in kinetic energy? (b) What is the ratio of the change in kinetic energy of car B to car A as seen by an observer moving with the initial velocity of car A?

Solution: (a) The ratio of the change in kinetic energy of car B to car A is

$$\frac{\Delta K_B}{\Delta K_A} = \frac{\frac{1}{2}m(v_{B,f})^2 - \frac{1}{2}m(v_{B,i})^2}{\frac{1}{2}m(v_{A,f})^2 - \frac{1}{2}m(v_{A,i})^2} = \frac{(v_{B,f})^2 - (v_{B,i})^2}{(v_{A,f})^2 - (v_{A,i})^2}$$
$$= \frac{(60\text{ mph})^2 - (50\text{ mph})^2}{(20\text{ mph})^2 - (10\text{ mph})^2} = 11/3.$$

Thus car B has a much greater increase in its kinetic energy than car A.

(b) In a reference moving with the speed of car A , car A increases its speed from rest to 10 mph and car B increases its speed from 40 to 50 mph. The ratio is now

$$\frac{\Delta K_B}{\Delta K_A} = \frac{\frac{1}{2}m(v_{B,f})^2 - \frac{1}{2}m(v_{B,0})^2}{\frac{1}{2}m(v_{A,f})^2 - \frac{1}{2}m(v_{A,0})^2} = \frac{(v_{B,f})^2 - (v_{B,0})^2}{(v_{A,f})^2 - (v_{A,0})^2}$$
$$= \frac{(50\text{ mph})^2 - (40\text{ mph})^2}{(10\text{ mph})^2} = 9.$$

The ratio is greater than that found in part a). Note that from the new reference frame both car A and car B have smaller increases in kinetic energy.

13.3 Kinematics and Kinetic Energy in One Dimension

13.3.1 Constant Accelerated Motion

Let's consider a constant accelerated motion of a *rigid body* in one dimension in which we treat the rigid body as a point mass. Suppose at $t = 0$ the body has an initial x-component of the velocity given by $v_{x,i}$. If the acceleration is in the direction of the displacement of the body then the body will increase its speed. If the acceleration is opposite the direction of the displacement then the acceleration will decrease the body's speed. The displacement of the body is given by

$$\Delta x = v_{x,i}\,t + \frac{1}{2}a_x t^2 . \tag{13.3.1}$$

The product of acceleration and the displacement is

$$a_x \Delta x = a_x (v_{x,i}\; t + \frac{1}{2}a_x t^2) . \tag{13.3.2}$$

The acceleration is given by

$$a_x = \frac{\Delta v_x}{\Delta t} = \frac{(v_{x,f} - v_{x,i})}{t}. \tag{13.3.3}$$

Therefore

$$a_x \Delta x = \frac{(v_{x,f} - v_{x,i})}{t}\left(v_{x,i}\; t + \frac{1}{2}\frac{(v_{x,f} - v_{x,i})}{t}t^2\right). \tag{13.3.4}$$

Equation (13.3.4) becomes

$$a_x \Delta x = (v_{x,f} - v_{x,i})(v_{x,i}) + \frac{1}{2}(v_{x,f} - v_{x,i})(v_{x,f} - v_{x,i}) = \frac{1}{2}v_{x,f}{}^2 - \frac{1}{2}v_{x,i}^2. \tag{13.3.5}$$

If we multiply each side of Equation (13.3.5) by the mass m of the object this kinematical result takes on an interesting interpretation for the motion of the object. We have

$$m\,a_x \Delta x = \frac{1}{2}m v_{x,f}^2 - m\frac{1}{2}v_{x,i}^2 = K_f - K_i. \tag{13.3.6}$$

Recall that for one-dimensional motion, Newton's Second Law is $F_x = m\,a_x$, for the motion considered here, Equation (13.3.6) becomes

$$F_x\,\Delta x = K_f - K_i. \tag{13.3.7}$$

13.3.2 Non-constant Accelerated Motion

If the acceleration is not constant, then we can divide the displacement into N intervals indexed by $j = 1$ to N. It will be convenient to denote the displacement intervals by Δx_j, the corresponding time intervals by Δt_j and the x-components of the velocities at the beginning and end of each interval as $v_{x,j-1}$ and $v_{x,j}$. Note that the x-component of the velocity at the beginning and end of the first interval $j = 1$ is then $v_{x,1} = v_{x,i}$ and the velocity at the end of the last interval, $j = N$ is $v_{x,N} = v_{x,j}$. Consider the sum of the products of the average acceleration $(a_{x,j})_{ave}$ and displacement Δx_j in each interval,

$$\sum_{j=1}^{j=N}(a_{x,j})_{\text{ave}}\Delta x_j. \tag{13.3.8}$$

The average acceleration over each interval is equal to

$$(a_{x,j})_{\text{ave}} = \frac{\Delta v_{x,j}}{\Delta t_j} = \frac{(v_{x,j+1} - v_{x,j})}{\Delta t_j}, \tag{13.3.9}$$

and so the contribution in each integral can be calculated as above and we have that

$$(a_{x,j})_{\text{ave}}\,\Delta x_j = \frac{1}{2}v_{x,j}{}^2 - \frac{1}{2}v_{x,j-1}{}^2\,. \tag{13.3.10}$$

When we sum over all the terms only the last and first terms survive, all the other terms cancel in pairs, and we have that

$$\sum_{j=1}^{j=N}(a_{x,j})_{\text{ave}}\,\Delta x_j = \frac{1}{2}v_{x,f}^2 - \frac{1}{2}v_{x,i}^2\,. \tag{13.3.11}$$

In the limit as $N \to \infty$ *and* $\Delta x_j \to 0$ for all j (both conditions must be met!), the limit of the sum is the definition of the definite integral of the acceleration with respect to the position,

$$\lim_{\substack{N\to\infty \\ \Delta x_j \to 0}} \sum_{j=1}^{j=N}(a_{x,j})_{\text{ave}}\,\Delta x_j \equiv \int_{x=x_i}^{x=x_f} a_x(x)\,dx\,. \tag{13.3.12}$$

Therefore In the limit as $N \to \infty$ *and* $\Delta x_j \to 0$ for all j, with $v_{x,N} \to v_{x,f}$, Eq. (13.3.11) becomes

$$\int_{x=x_i}^{x=x_f} a_x(x)\,dx = \frac{1}{2}(v_{x,f}^2 - v_{x,i}^2) \tag{13.3.13}$$

This integral result is consequence of the definition that $a_x \equiv dv_x / dt$. The integral in Eq. (13.3.13) is an integral with respect to space, while our previous integral

$$\int_{t=t_i}^{t=t_f} a_x(t)\,dt = v_{x,f} - v_{x,i}\,. \tag{13.3.14}$$

requires integrating acceleration with respect to time. Multiplying both sides of Eq. (13.3.13) by the mass m yields

$$\int_{x=x_i}^{x=x_f} m a_x(x)\,dx = \frac{1}{2}m(v_{x,f}^2 - v_{x,i}^2) = K_f - K_i\,. \tag{13.3.15}$$

When we introduce Newton's Second Law in the form $F_x = m a_x$, then Eq. (13.3.15) becomes

$$\int_{x=x_i}^{x=x_f} F_x(x)\,dx = K_f - K_i. \tag{13.3.16}$$

The integral of the x-component of the force with respect to displacement in Eq. (13.3.16) applies to the motion of a point-like object. For extended bodies, Eq. (13.3.16) applies to the center of mass motion because the external force on a rigid body causes the center of mass to accelerate.