17.1 Introduction

A body is called a *rigid body* if the distance between any two points in the body does not change in time. Rigid bodies, unlike point masses, can have forces applied at different points in the body. For most objects, treating as a rigid body is an idealization, but a very good one. In addition to forces applied at points, forces may be distributed over the entire body. Forces that are distributed over a body are difficult to analyze; however, for example, we regularly experience the effect of the gravitational force on bodies. Based on our experience observing the effect of the gravitational force on rigid bodies, we shall demonstrate that the gravitational force can be concentrated at a point in the rigid body called the *center of gravity*, which for small bodies (so that $\vec{\mathbf{g}}$ may be taken as constant within the body) is identical to the *center of mass* of the body.

Let's consider a rigid rod thrown in the air (Figure 17.1) so that the rod is spinning as its center of mass moves with velocity $\vec{\mathbf{v}}_{\text{cm}}$. We have explored the physics of translational motion; now, we wish to investigate the properties of rotational motion exhibited in the rod's motion, beginning with the notion that every particle is rotating about the center of mass with the same angular (rotational) velocity.

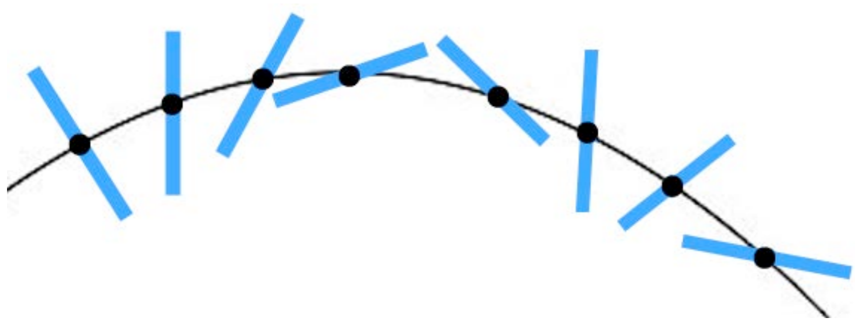

Figure 17.1 The center of mass of a thrown rigid rod follows a parabolic trajectory while the rod rotates about the center of mass.

We can use Newton's Second Law to predict how the center of mass will move. Because the only external force on the rod is the gravitational force (neglecting the action of air resistance), the center of mass of the body will move in a parabolic trajectory.

How was the rod induced to rotate? In order to spin the rod, we applied a torque with our fingers and wrist to one end of the rod as the rod was released. The applied torque is proportional to the angular acceleration. The constant of proportionality is the moment of inertia. When external forces and torques are present, the motion of a rigid body can be extremely complicated while it is translating and rotating in space.

In order to describe the relationship between torque, moment of inertia, and angular acceleration, we will introduce a new vector operation called the ***vector product*** also know as the "cross product" that takes any two vectors and generates a new vector. The vector product is a type of "multiplication" law that turns our vector space (law for addition of vectors) into a vector algebra (a vector algebra is a vector space with an additional rule for multiplication of vectors).

17.2 Vector Product (Cross Product)

Let $\vec{\mathbf{A}}$ *and* $\vec{\mathbf{B}}$ *be two vectors. Because any two non-parallel vectors form a plane, we denote the angle* θ *to be the angle between the vectors* $\vec{\mathbf{A}}$ *and* $\vec{\mathbf{B}}$ *as shown in Figure 17.2. The* ***magnitude of the vector product*** $\vec{\mathbf{A}}\times\vec{\mathbf{B}}$ *of the vectors* $\vec{\mathbf{A}}$ *and* $\vec{\mathbf{B}}$ *is defined to be product of the magnitude of the vectors* $\vec{\mathbf{A}}$ *and* $\vec{\mathbf{B}}$ *with the sine of the angle* θ *between the two vectors,*

$$\left|\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right| = \left|\vec{\mathbf{A}}\right|\left|\vec{\mathbf{B}}\right|\sin(\theta). \qquad (17.2.1)$$

The angle θ *between the vectors is limited to the values* $0\le\theta\le\pi$ *ensuring that* $\sin(\theta)\ge 0$.

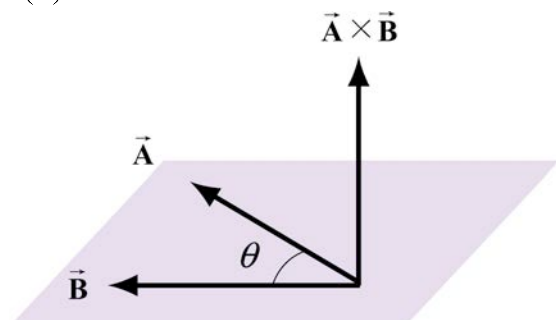

Figure 17.2 Vector product geometry.

The direction of the vector product is defined as follows. The vectors $\vec{\mathbf{A}}$ *and* $\vec{\mathbf{B}}$ *form a plane. Consider the direction perpendicular to this plane. There are two possibilities: we shall choose one of these two (the one shown in Figure 17.2) for the direction of the vector product* $\vec{\mathbf{A}}\times\vec{\mathbf{B}}$ *using a convention that is commonly called the "****right-hand rule****".*

17.2.1 Right-hand Rule for the Direction of Vector Product

The first step is to redraw the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ so that the tails are touching. Then draw an arc starting from the vector $\vec{\mathbf{A}}$ and finishing on the vector $\vec{\mathbf{B}}$. Curl your right fingers the same way as the arc. Your right thumb points in the direction of the vector product $\vec{\mathbf{A}}\times\vec{\mathbf{B}}$ (Figure 17.3).

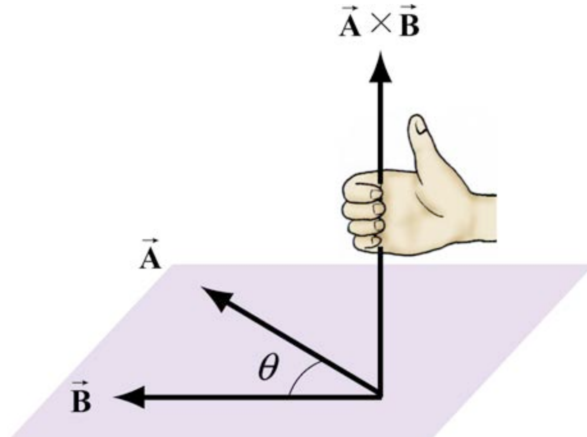

Figure 17.3 Right-Hand Rule.

You should remember that the direction of the vector product $\vec{\mathbf{A}}\times\vec{\mathbf{B}}$ is perpendicular to the plane formed by $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$. We can give a geometric interpretation to the magnitude of the vector product by writing the magnitude as

$$\left|\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right| = \left|\vec{\mathbf{A}}\right|\left(\left|\vec{\mathbf{B}}\right|\sin\theta\right). \tag{17.2.2}$$

The vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ form a parallelogram. The area of the parallelogram is equal to the height times the base, which is the magnitude of the vector product. In Figure 17.4, two different representations of the height and base of a parallelogram are illustrated. As depicted in Figure 17.4a, the term $\left|\vec{\mathbf{B}}\right|\sin\theta$ is the projection of the vector $\vec{\mathbf{B}}$ in the direction perpendicular to the vector $\vec{\mathbf{B}}$. We could also write the magnitude of the vector product as

$$\left|\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right| = \left(\left|\vec{\mathbf{A}}\right|\sin\theta\right)\left|\vec{\mathbf{B}}\right|. \tag{17.2.3}$$

The term $\left|\vec{\mathbf{A}}\right|\sin\theta$ is the projection of the vector $\vec{\mathbf{A}}$ in the direction perpendicular to the vector $\vec{\mathbf{B}}$ as shown in Figure 17.4(b). The vector product of two vectors that are parallel (or anti-parallel) to each other is zero because the angle between the vectors is 0 (or π) and $\sin(0)=0$ (or $\sin(\pi)=0$). Geometrically, two parallel vectors do not have a unique component perpendicular to their common direction.

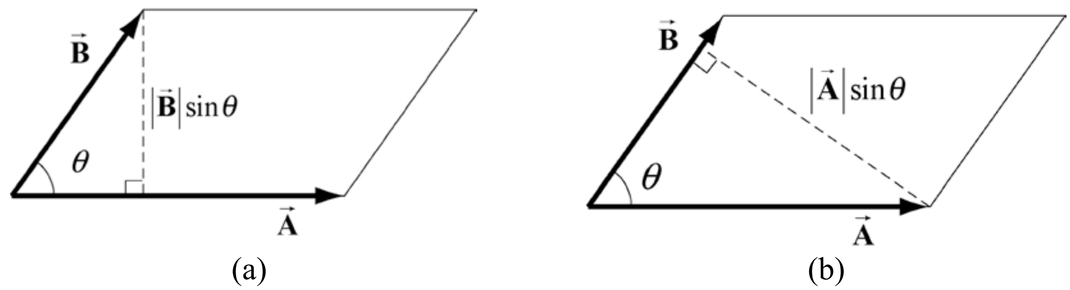

Figure 17.4 Projection of (a) $\vec{\mathbf{B}}$ perpendicular to $\vec{\mathbf{A}}$, (b) of $\vec{\mathbf{A}}$ perpendicular to $\vec{\mathbf{B}}$

17.2.2 Properties of the Vector Product

(1) The vector product is anti-commutative because changing the order of the vectors changes the direction of the vector product by the right hand rule:

$$\vec{\mathbf{A}}\times\vec{\mathbf{B}} = -\vec{\mathbf{B}}\times\vec{\mathbf{A}}\,. \qquad (17.2.4)$$

(2) The vector product between a vector $c\,\vec{\mathbf{A}}$ where c is a scalar and a vector $\vec{\mathbf{B}}$ is

$$c\,\vec{\mathbf{A}}\times\vec{\mathbf{B}} = c\,(\vec{\mathbf{A}}\times\vec{\mathbf{B}})\,. \qquad (17.2.5)$$

Similarly,

$$\vec{\mathbf{A}}\times c\,\vec{\mathbf{B}} = c\,(\vec{\mathbf{A}}\times\vec{\mathbf{B}})\,. \qquad (17.2.6)$$

(3) The vector product between the sum of two vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ with a vector $\vec{\mathbf{C}}$ is

$$(\vec{\mathbf{A}}+\vec{\mathbf{B}})\times\vec{\mathbf{C}} = \vec{\mathbf{A}}\times\vec{\mathbf{C}}+\vec{\mathbf{B}}\times\vec{\mathbf{C}} \qquad (17.2.7)$$

Similarly,

$$\vec{\mathbf{A}}\times(\vec{\mathbf{B}}+\vec{\mathbf{C}}) = \vec{\mathbf{A}}\times\vec{\mathbf{B}}+\vec{\mathbf{A}}\times\vec{\mathbf{C}}\,. \qquad (17.2.8)$$

17.2.3 Vector Decomposition and the Vector Product: Cartesian Coordinates

We first calculate that the magnitude of vector product of the unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$:

$$|\,\hat{\mathbf{i}}\times\hat{\mathbf{j}}\,| = |\,\hat{\mathbf{i}}\,||\,\hat{\mathbf{j}}\,|\sin(\pi/2) = 1\,, \qquad (17.2.9)$$

because the unit vectors have magnitude $|\,\hat{\mathbf{i}}\,| = |\,\hat{\mathbf{j}}\,| = 1$ and $\sin(\pi/2) = 1$. By the right hand rule, the direction of $\hat{\mathbf{i}}\times\hat{\mathbf{j}}$ is in the $+\hat{\mathbf{k}}$ as shown in Figure 17.5. Thus $\hat{\mathbf{i}}\times\hat{\mathbf{j}} = \hat{\mathbf{k}}$.

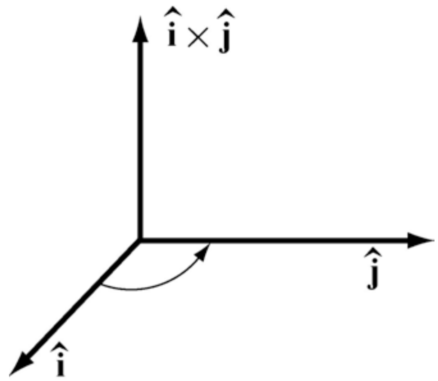

Figure 17.5 Vector product of $\hat{\mathbf{i}}\times\hat{\mathbf{j}}$

We note that the same rule applies for the unit vectors in the *y* and *z* directions,

$$\hat{\mathbf{j}}\times\hat{\mathbf{k}}=\hat{\mathbf{i}},\quad \hat{\mathbf{k}}\times\hat{\mathbf{i}}=\hat{\mathbf{j}}\,. \qquad (17.2.10)$$

By the anti-commutatively property (1) of the vector product,

$$\hat{\mathbf{j}}\times\hat{\mathbf{i}}=-\hat{\mathbf{k}},\quad \hat{\mathbf{i}}\times\hat{\mathbf{k}}=-\hat{\mathbf{j}} \qquad (17.2.11)$$

The vector product of the unit vector $\hat{\mathbf{i}}$ with itself is zero because the two unit vectors are parallel to each other, ($\sin(0)=0$),

$$|\,\hat{\mathbf{i}}\times\hat{\mathbf{i}}\,|=|\,\hat{\mathbf{i}}\,||\,\hat{\mathbf{i}}\,|\sin(0)=0\,. \qquad (17.2.12)$$

The vector product of the unit vector $\hat{\mathbf{j}}$ with itself and the unit vector $\hat{\mathbf{k}}$ with itself are also zero for the same reason,

$$\left|\hat{\mathbf{j}}\times\hat{\mathbf{j}}\right|=0,\quad \left|\hat{\mathbf{k}}\times\hat{\mathbf{k}}\right|=0\,. \qquad (17.2.13)$$

With these properties in mind we can now develop an algebraic expression for the vector product in terms of components. Let's choose a Cartesian coordinate system with the vector $\vec{\mathbf{B}}$ pointing along the positive *x*-axis with positive *x*-component B_x. Then the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ can be written as

$$\vec{\mathbf{A}}=A_x\,\hat{\mathbf{i}}+A_y\,\hat{\mathbf{j}}+A_z\,\hat{\mathbf{k}} \qquad (17.2.14)$$

$$\vec{\mathbf{B}}=B_x\,\hat{\mathbf{i}}\,, \qquad (17.2.15)$$

respectively. The vector product in vector components is

$$\vec{\mathbf{A}}\times\vec{\mathbf{B}}=(A_x\,\hat{\mathbf{i}}+A_y\,\hat{\mathbf{j}}+A_z\,\hat{\mathbf{k}})\times B_x\,\hat{\mathbf{i}}\,. \qquad (17.2.16)$$

This becomes,

$$
\begin{aligned}
\vec{\mathbf{A}}\times\vec{\mathbf{B}} &= (A_x\ \hat{\mathbf{i}}\times B_x\ \hat{\mathbf{i}}) + (A_y\ \hat{\mathbf{j}}\times B_x\ \hat{\mathbf{i}}) + (A_z\ \hat{\mathbf{k}}\times B_x\ \hat{\mathbf{i}}) \\
&= A_x B_x\ (\hat{\mathbf{i}}\times\hat{\mathbf{i}}) + A_y B_x\ (\hat{\mathbf{j}}\times\hat{\mathbf{i}}) + A_z B_x\ (\hat{\mathbf{k}}\times\hat{\mathbf{i}}) \quad . \\
&= -A_y B_x\ \hat{\mathbf{k}} + A_z B_x\ \hat{\mathbf{j}}
\end{aligned}
\tag{17.2.17}
$$

The vector component expression for the vector product easily generalizes for arbitrary vectors

$$\vec{\mathbf{A}} = A_x\ \hat{\mathbf{i}} + A_y\ \hat{\mathbf{j}} + A_z\ \hat{\mathbf{k}} \tag{17.2.18}$$

$$\vec{\mathbf{B}} = B_x\ \hat{\mathbf{i}} + B_y\ \hat{\mathbf{j}} + B_z\ \hat{\mathbf{k}}, \tag{17.2.19}$$

to yield

$$\vec{\mathbf{A}}\times\vec{\mathbf{B}} = (A_y B_z - A_z B_y)\ \hat{\mathbf{i}} + (A_z B_x - A_x B_z)\ \hat{\mathbf{j}} + (A_x B_y - A_y B_x)\ \hat{\mathbf{k}}\ . \tag{17.2.20}$$

17.2.4 Vector Decomposition and the Vector Product: Cylindrical Coordinates

Recall the cylindrical coordinate system, which we show in Figure 17.6. We have chosen two directions, radial and tangential in the plane, and a perpendicular direction to the plane.

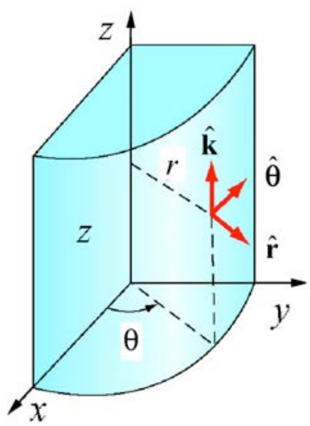

Figure 17.6 Cylindrical coordinates

The unit vectors are at right angles to each other and so using the right hand rule, the vector product of the unit vectors are given by the relations

$$\hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}} = \hat{\mathbf{k}} \tag{17.2.21}$$

$$\hat{\boldsymbol{\theta}}\times\hat{\mathbf{k}} = \hat{\mathbf{r}} \tag{17.2.22}$$

$$\hat{\mathbf{k}}\times\hat{\mathbf{r}} = \hat{\boldsymbol{\theta}}\ . \tag{17.2.23}$$

Because the vector product satisfies $\vec{\mathbf{A}}\times\vec{\mathbf{B}} = -\vec{\mathbf{B}}\times\vec{\mathbf{A}}$, we also have that

$$\hat{\boldsymbol{\theta}}\times\hat{\mathbf{r}} = -\hat{\mathbf{k}} \tag{17.2.24}$$

$$\hat{\mathbf{k}}\times\hat{\boldsymbol{\theta}} = -\hat{\mathbf{r}} \tag{17.2.25}$$

$$\hat{\mathbf{r}}\times\hat{\mathbf{k}} = -\hat{\boldsymbol{\theta}}\,. \tag{17.2.26}$$

Finally

$$\hat{\mathbf{r}}\times\hat{\mathbf{r}} = \hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\theta}} = \hat{\mathbf{k}}\times\hat{\mathbf{k}} = \vec{\mathbf{0}}\,. \tag{17.2.27}$$

Example 17.1 Vector Products

Given two vectors, $\vec{\mathbf{A}} = 2\,\hat{\mathbf{i}} + -3\,\hat{\mathbf{j}} + 7\,\hat{\mathbf{k}}$ and $\vec{\mathbf{B}} = 5\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$, find $\vec{\mathbf{A}}\times\vec{\mathbf{B}}$.

Solution:

$$\begin{aligned}\vec{\mathbf{A}}\times\vec{\mathbf{B}} &= (A_y B_z - A_z B_y)\,\hat{\mathbf{i}} + (A_z B_x - A_x B_z)\,\hat{\mathbf{j}} + (A_x B_y - A_y B_x)\,\hat{\mathbf{k}} \\ &= ((-3)(2) - (7)(1))\,\hat{\mathbf{i}} + ((7)(5) - (2)(2))\,\hat{\mathbf{j}} + ((2)(1) - (-3)(5))\,\hat{\mathbf{k}} \\ &= -13\,\hat{\mathbf{i}} + 31\,\hat{\mathbf{j}} + 17\,\hat{\mathbf{k}}.\end{aligned}$$

Example 17.2 Law of Sines

For the triangle shown in Figure 17.7a, prove the law of sines, $\left|\vec{\mathbf{A}}\right| / \sin\alpha = \left|\vec{\mathbf{B}}\right| / \sin\beta = \left|\vec{\mathbf{C}}\right| / \sin\gamma$, using the vector product.

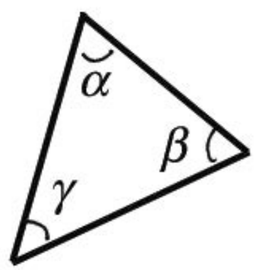

Figure 17.7 (a) Example 17.2

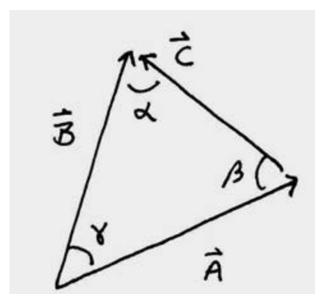

Figure 17.7 (b) Vector analysis

Solution: Consider the area of a triangle formed by three vectors $\vec{\mathbf{A}}$, $\vec{\mathbf{B}}$, and $\vec{\mathbf{C}}$, where $\vec{\mathbf{A}} + \vec{\mathbf{B}} + \vec{\mathbf{C}} = 0$ (Figure 17.7b). Because $\vec{\mathbf{A}} + \vec{\mathbf{B}} + \vec{\mathbf{C}} = 0$, we have that $0 = \vec{\mathbf{A}}\times(\vec{\mathbf{A}} + \vec{\mathbf{B}} + \vec{\mathbf{C}}) = \vec{\mathbf{A}}\times\vec{\mathbf{B}} + \vec{\mathbf{A}}\times\vec{\mathbf{C}}$. Thus $\vec{\mathbf{A}}\times\vec{\mathbf{B}} = -\vec{\mathbf{A}}\times\vec{\mathbf{C}}$ or $\left|\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right| = \left|\vec{\mathbf{A}}\times\vec{\mathbf{C}}\right|$. From Figure 17.7b we see that $\left|\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right| = \left|\vec{\mathbf{A}}\right|\left|\vec{\mathbf{B}}\right|\sin\gamma$ and $\left|\vec{\mathbf{A}}\times\vec{\mathbf{C}}\right| = \left|\vec{\mathbf{A}}\right|\left|\vec{\mathbf{C}}\right|\sin\beta$. Therefore $\left|\vec{\mathbf{A}}\right|\left|\vec{\mathbf{B}}\right|\sin\gamma = \left|\vec{\mathbf{A}}\right|\left|\vec{\mathbf{C}}\right|\sin\beta$, and hence $\left|\vec{\mathbf{B}}\right| / \sin\beta = \left|\vec{\mathbf{C}}\right| / \sin\gamma$. A similar argument shows that $\left|\vec{\mathbf{B}}\right| / \sin\beta = \left|\vec{\mathbf{A}}\right| / \sin\alpha$ proving the law of sines.

Example 17.3 Unit Normal

Find a unit vector perpendicular to $\vec{\mathbf{A}} = \hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $\vec{\mathbf{B}} = -2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$.

Solution: The vector product $\vec{\mathbf{A}}\times\vec{\mathbf{B}}$ is perpendicular to both $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$. Therefore the unit vectors $\hat{\mathbf{n}} = \pm\vec{\mathbf{A}}\times\vec{\mathbf{B}}/\left|\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right|$ are perpendicular to both $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$. We first calculate

$$\begin{aligned}\vec{\mathbf{A}}\times\vec{\mathbf{B}} &= (A_yB_z - A_zB_y)\,\hat{\mathbf{i}} + (A_zB_x - A_xB_z)\,\hat{\mathbf{j}} + (A_xB_y - A_yB_x)\,\hat{\mathbf{k}}\\ &= ((1)(3)-(-1)(-1))\,\hat{\mathbf{i}} + ((-1)(2)-(1)(3))\,\hat{\mathbf{j}} + ((1)(-1)-(1)(2))\,\hat{\mathbf{k}}\\ &= 2\,\hat{\mathbf{i}} - 5\,\hat{\mathbf{j}} - 3\,\hat{\mathbf{k}}.\end{aligned}$$

We now calculate the magnitude

$$\left|\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right| = (2^2+5^2+3^2)^{1/2} = (38)^{1/2} \ .$$

Therefore the perpendicular unit vectors are

$$\hat{\mathbf{n}} = \pm\vec{\mathbf{A}}\times\vec{\mathbf{B}}/\left|\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right| = \pm(2\,\hat{\mathbf{i}} - 5\,\hat{\mathbf{j}} - 3\,\hat{\mathbf{k}})/(38)^{1/2}.$$

Example 17.4 Volume of Parallelepiped

Show that the volume of a parallelepiped with edges formed by the vectors $\vec{\mathbf{A}}$, $\vec{\mathbf{B}}$, and $\vec{\mathbf{C}}$ is given by $\vec{\mathbf{A}}\cdot(\vec{\mathbf{B}}\times\vec{\mathbf{C}})$.

Solution: The volume of a parallelepiped is given by area of the base times height. If the base is formed by the vectors $\vec{\mathbf{B}}$ and $\vec{\mathbf{C}}$, then the area of the base is given by the magnitude of $\vec{\mathbf{B}}\times\vec{\mathbf{C}}$. The vector $\vec{\mathbf{B}}\times\vec{\mathbf{C}} = \left|\vec{\mathbf{B}}\times\vec{\mathbf{C}}\right|\hat{\mathbf{n}}$ where $\hat{\mathbf{n}}$ is a unit vector perpendicular to the base (Figure 17.8).

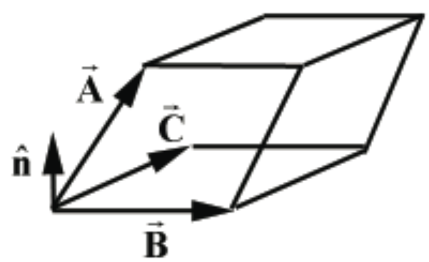

Figure 17.8 Example 17.4

The projection of the vector $\vec{\mathbf{A}}$ along the direction $\hat{\mathbf{n}}$ gives the height of the parallelepiped. This projection is given by taking the dot product of $\vec{\mathbf{A}}$ with a unit vector and is equal to $\vec{\mathbf{A}}\cdot\hat{\mathbf{n}} = \textit{height}$. Therefore

$$\vec{\mathbf{A}}\cdot(\vec{\mathbf{B}}\times\vec{\mathbf{C}}) = \vec{\mathbf{A}}\cdot(\left|\vec{\mathbf{B}}\times\vec{\mathbf{C}}\right|)\hat{\mathbf{n}} = (\left|\vec{\mathbf{B}}\times\vec{\mathbf{C}}\right|)\vec{\mathbf{A}}\cdot\hat{\mathbf{n}} = (\textit{area})(\textit{height}) = (\textit{volume}).$$

Example 17.5 Vector Decomposition

Let $\vec{\mathbf{A}}$ be an arbitrary vector and let $\hat{\mathbf{n}}$ be a unit vector in some fixed direction. Show that $\vec{\mathbf{A}} = (\vec{\mathbf{A}} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} + (\hat{\mathbf{n}} \times \vec{\mathbf{A}}) \times \hat{\mathbf{n}}$.

Solution: Let $\vec{\mathbf{A}} = A_{\parallel}\hat{\mathbf{n}} + A_{\perp}\hat{\mathbf{e}}$ where $A_{\parallel}$ is the component $\vec{\mathbf{A}}$ in the direction of $\hat{\mathbf{n}}$, $\hat{\mathbf{e}}$ is the direction of the projection of $\vec{\mathbf{A}}$ in a plane perpendicular to $\hat{\mathbf{n}}$, and $A_{\perp}$ is the component of $\vec{\mathbf{A}}$ in the direction of $\hat{\mathbf{e}}$. Because $\hat{\mathbf{e}} \cdot \hat{\mathbf{n}} = 0$, we have that $\vec{\mathbf{A}} \cdot \hat{\mathbf{n}} = A_{\parallel}$. Note that

$$\hat{\mathbf{n}} \times \vec{\mathbf{A}} = \hat{\mathbf{n}} \times (A_{\parallel}\hat{\mathbf{n}} + A_{\perp}\hat{\mathbf{e}}) = \hat{\mathbf{n}} \times A_{\perp}\hat{\mathbf{e}} = A_{\perp}(\hat{\mathbf{n}} \times \hat{\mathbf{e}}).$$

The unit vector $\hat{\mathbf{n}} \times \hat{\mathbf{e}}$ lies in the plane perpendicular to $\hat{\mathbf{n}}$ and is also perpendicular to $\hat{\mathbf{e}}$. Therefore $(\hat{\mathbf{n}} \times \hat{\mathbf{e}}) \times \hat{\mathbf{n}}$ is also a unit vector that is parallel to $\hat{\mathbf{e}}$ (by the right hand rule. So $(\hat{\mathbf{n}} \times \vec{\mathbf{A}}) \times \hat{\mathbf{n}} = A_{\perp}\hat{\mathbf{e}}$. Thus

$$\vec{\mathbf{A}} = A_{\parallel}\hat{\mathbf{n}} + A_{\perp}\hat{\mathbf{e}} = (\vec{\mathbf{A}} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} + (\hat{\mathbf{n}} \times \vec{\mathbf{A}}) \times \hat{\mathbf{n}}.$$

17.3 Torque

17.3.1 Definition of Torque about a Point

In order to understand the dynamics of a rotating rigid body we will introduce a new quantity, the torque. Let a force $\vec{\mathbf{F}}_P$ with magnitude $F = |\vec{\mathbf{F}}_P|$ act at a point P. Let $\vec{\mathbf{r}}_{S,P}$ be the vector from the point S to a point P, with magnitude $r = |\vec{\mathbf{r}}_{S,P}|$. The angle between the vectors $\vec{\mathbf{r}}_{S,P}$ and $\vec{\mathbf{F}}_P$ is θ with $[0 \le \theta \le \pi]$ (Figure 17.9).

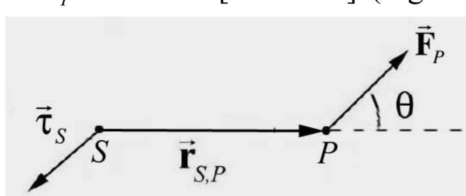

Figure 17.9 Torque about a point S due to a force acting at a point P

The torque about a point S due to force $\vec{\mathbf{F}}_P$ acting at P, is defined by

$$\vec{\boldsymbol{\tau}}_S = \vec{\mathbf{r}}_{S,P} \times \vec{\mathbf{F}}_P. \qquad (17.2.28)$$

The magnitude of the torque about a point S due to force $\vec{\mathbf{F}}_P$ acting at P, is given by

$$\tau_S \equiv \left|\vec{\boldsymbol{\tau}}_S\right| = r\,F\sin\theta\,. \tag{17.2.29}$$

The SI units for torque are $[\mathrm{N}\cdot\mathrm{m}]$. The direction of the torque is perpendicular to the plane formed by the vectors $\vec{\mathbf{r}}_{S,P}$ and $\vec{\mathbf{F}}_P$ (for $[0<\theta<\pi]$), and by definition points in the direction of the unit normal vector to the plane $\hat{\mathbf{n}}_{RHR}$ as shown in Figure 17.10.

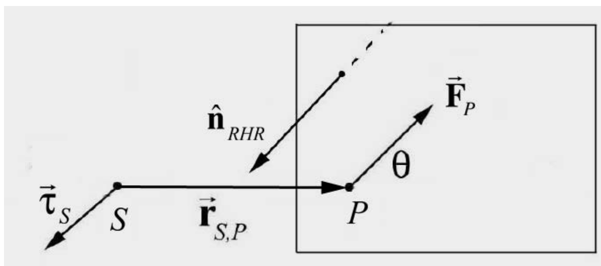

Figure 17.10 Vector direction for the torque

Figure 17.11 shows the two different ways of defining height and base for a parallelogram defined by the vectors $\vec{\mathbf{r}}_{S,P}$ and $\vec{\mathbf{F}}_P$.

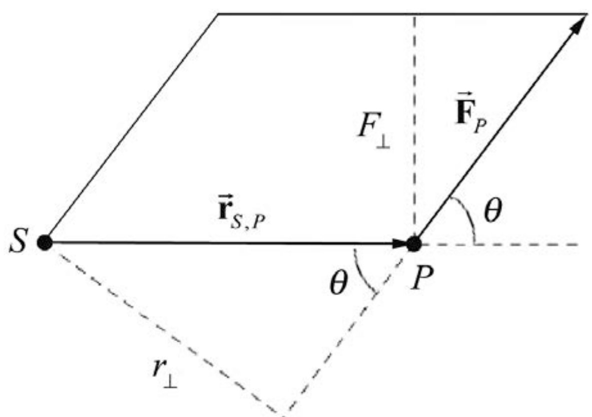

Figure 17.11 Area of the torque parallelogram.

Let $r_\perp = r\sin\theta$ and let $F_\perp = F\sin\theta$ be the component of the force $\vec{\mathbf{F}}_P$ that is perpendicular to the line passing from the point S to P. (Recall the angle θ has a range of values $0 \le \theta \le \pi$ so both $r_\perp \ge 0$ and $F_\perp \ge 0$.) Then the area of the parallelogram defined by $\vec{\mathbf{r}}_{S,P}$ and $\vec{\mathbf{F}}_P$ is given by

$$\text{Area} = \tau_S = r_\perp F = r\,F_\perp = r\,F\sin\theta\,. \tag{17.2.30}$$

We can interpret the quantity $r_\perp$ as follows.

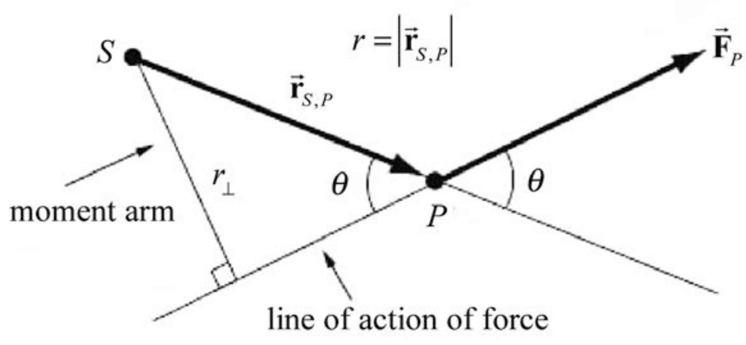

Figure 17.12 The moment arm about the point S and line of action of force passing through the point P

We begin by drawing the ***line of action of the force*** $\vec{\mathbf{F}}_P$. This is a straight line passing through P, parallel to the direction of the force $\vec{\mathbf{F}}_P$. Draw a perpendicular to this line of action that passes through the point S (Figure 17.12). The length of this perpendicular, $r_\perp = r\sin\theta$, is called ***the moment arm about the point S of the force*** $\vec{\mathbf{F}}_P$.

You should keep in mind three important properties of torque:

1. The torque is zero if the vectors $\vec{\mathbf{r}}_{S,P}$ and $\vec{\mathbf{F}}_P$ are parallel $(\theta = 0)$ or anti-parallel $(\theta = \pi)$.

2. Torque is a vector whose direction and magnitude depend on the choice of a point S about which the torque is calculated.

3. The direction of torque is perpendicular to the plane formed by the two vectors, $\vec{\mathbf{F}}_P$ and $r = |\vec{\mathbf{r}}_{S,P}|$ (the vector from the point S to a point P).

17.3.2 Alternative Approach to Assigning a Sign Convention for Torque

In the case where all of the forces $\vec{\mathbf{F}}_i$ and position vectors $\vec{\mathbf{r}}_{i,P}$ are coplanar (or zero), we can, instead of referring to the direction of torque, assign a purely algebraic positive or negative sign to torque according to the following convention. We note that the arc in Figure 17.13a circles in counterclockwise direction. (Figures 17.13a and 17.13b use the simplifying assumption, for the purpose of the figure only, that the two vectors in question, $\vec{\mathbf{F}}_P$ and $\vec{\mathbf{r}}_{S,P}$ are perpendicular. The point S about which torques are calculated is not shown.)

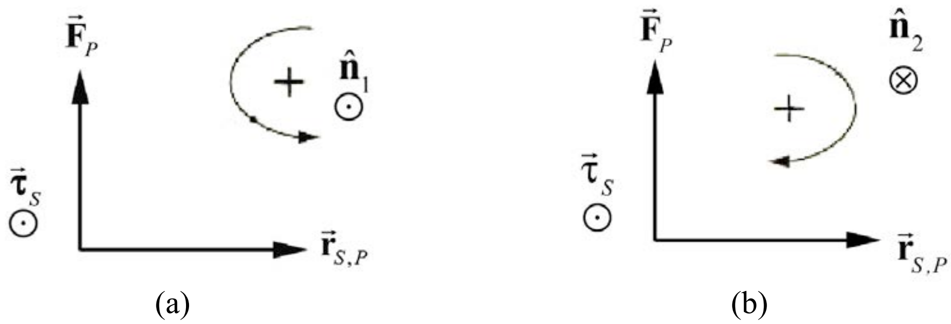

Figure 17.13 (a) Positive torque out of plane, **(b)** positive torque into plane

We can associate with this counterclockwise orientation a unit normal vector $\hat{\mathbf{n}}$ according to the right-hand rule: curl your right hand fingers in the counterclockwise direction and your right thumb will then point in the $\hat{\mathbf{n}}_1$ direction (Figure 17.13a). The arc in Figure 17.13b circles in the clockwise direction, and we associate this orientation with the unit normal $\hat{\mathbf{n}}_2$.

It's important to note that the terms "clockwise" and "counterclockwise" might be different for different observers. For instance, if the plane containing $\vec{\mathbf{F}}_P$ and $\vec{\mathbf{r}}_{S,P}$ is horizontal, an observer above the plane and an observer below the plane would disagree on the two terms. For a vertical plane, the directions that two observers on opposite sides of the plane would be mirror images of each other, and so again the observers would disagree.

1. Suppose we choose counterclockwise as positive. Then we assign a positive sign for the component of the torque when the torque is in the same direction as the unit normal $\hat{\mathbf{n}}_1$, i.e. $\vec{\boldsymbol{\tau}}_S = \vec{\mathbf{r}}_{S,P} \times \vec{\mathbf{F}}_P = +\left|\vec{\mathbf{r}}_{S,P}\right|\left|\vec{\mathbf{F}}_P\right|\hat{\mathbf{n}}_1$, (Figure 17.13a).

2. Suppose we choose clockwise as positive. Then we assign a negative sign for the component of the torque in Figure 17.13b because the torque is directed opposite to the unit normal $\hat{\mathbf{n}}_2$, i.e. $\vec{\boldsymbol{\tau}}_S = \vec{\mathbf{r}}_{S,P} \times \vec{\mathbf{F}}_P = -\left|\vec{\mathbf{r}}_{S,P}\right|\left|\vec{\mathbf{F}}_P\right|\hat{\mathbf{n}}_2$.

Example 17.6 Torque and Vector Product

Consider two vectors $\vec{\mathbf{r}}_{P,F} = x\hat{\mathbf{i}}$ with $x > 0$ and $\vec{\mathbf{F}} = F_x\hat{\mathbf{i}} + F_z\hat{\mathbf{k}}$ with $F_x > 0$ and $F_z > 0$. Calculate the torque $\vec{\mathbf{r}}_{P,F} \times \vec{\mathbf{F}}$.

Solution: We calculate the vector product noting that in a right handed choice of unit vectors, $\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \vec{\mathbf{0}}$ and $\hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}}$,

$$\vec{\mathbf{r}}_{P,F} \times \vec{\mathbf{F}} = x\hat{\mathbf{i}} \times (F_x\hat{\mathbf{i}} + F_z\hat{\mathbf{k}}) = (x\hat{\mathbf{i}} \times F_x\hat{\mathbf{i}}) + (x\hat{\mathbf{i}} \times F_z\hat{\mathbf{k}}) = -xF_z\,\hat{\mathbf{j}}.$$

Because $x > 0$ and $F_z > 0$, the direction of the vector product is in the negative y-direction.

Example 17.7 Calculating Torque

In Figure 17.14, a force of magnitude F is applied to one end of a lever of length L. What is the magnitude and direction of the torque about the point S?

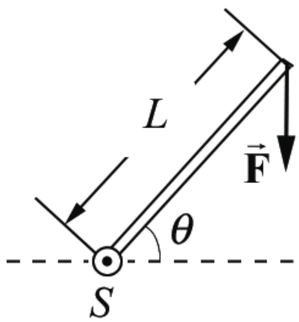

Figure 17.14 Example 17.7

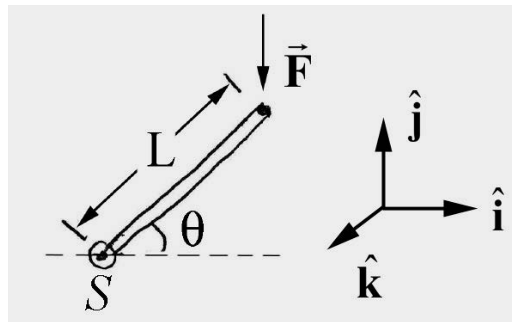

Figure 17.15 Coordinate system

Solution: Choose units vectors such that $\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$, with $\hat{\mathbf{i}}$ pointing to the right and $\hat{\mathbf{j}}$ pointing up (Figure 17.15). The torque about the point S is given by $\vec{\boldsymbol{\tau}}_S = \vec{\mathbf{r}}_{S,F} \times \vec{\mathbf{F}}$, where $\vec{\mathbf{r}}_{SF} = L\cos\theta\hat{\mathbf{i}} + L\sin\theta\hat{\mathbf{j}}$ and $\vec{\mathbf{F}} = -F\hat{\mathbf{j}}$ then

$$\vec{\boldsymbol{\tau}}_S = (L\cos\theta\,\hat{\mathbf{i}} + L\sin\theta\,\hat{\mathbf{j}}) \times -F\,\hat{\mathbf{j}} = -FL\cos\theta\,\hat{\mathbf{k}}\,.$$

Example 17.8 Torque and the Ankle

A person of mass m is crouching with their weight evenly distributed on both tiptoes. The free-body force diagram on the skeletal part of the foot is shown in Figure 17.16. The normal force $\vec{\mathbf{N}}$ acts at the contact point between the foot and the ground. In this position, the tibia acts on the foot at the point S with a force $\vec{\mathbf{F}}$ of an unknown magnitude $F = |\vec{\mathbf{F}}|$ and makes an unknown angle β with the vertical. This force acts on the ankle a horizontal distance s from the point where the foot contacts the floor. The Achilles tendon also acts on the foot and is under considerable tension with magnitude $T \equiv |\vec{\mathbf{T}}|$ and acts at an angle α with the horizontal as shown in the figure. The tendon acts on the ankle a horizontal distance b from the point S where the tibia acts on the foot. You may ignore the weight of the foot. Let g be the gravitational constant. Compute the torque about the point S due to (a) the tendon force on the foot; (b) the force of the tibia on the foot; (c) the normal force of the floor on the foot.

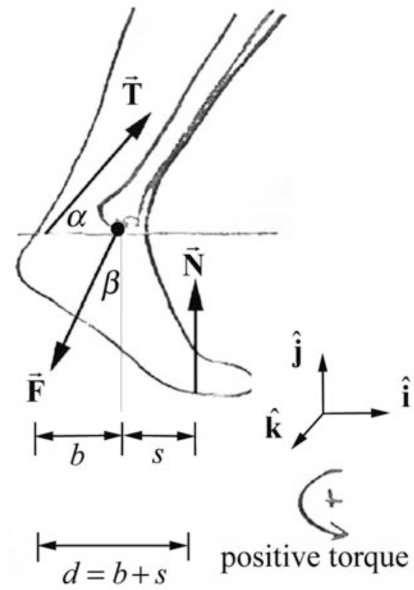

Figure 17.16 Force diagram and coordinate system for ankle

Solution: (a) We shall first calculate the torque due to the force of the Achilles tendon on the ankle. The tendon force has the vector decomposition $\vec{\mathbf{T}} = T\cos\alpha\,\hat{\mathbf{i}} + T\sin\alpha\,\hat{\mathbf{j}}$.

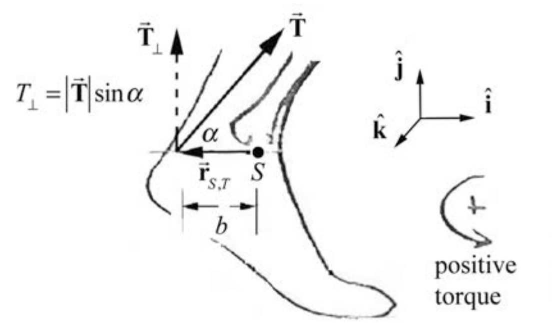

Figure 17.17 Torque diagram for tendon force on ankle

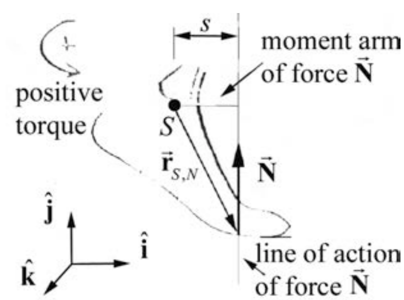

Figure 17.18 Torque diagram for normal force on ankle $\vec{\mathbf{r}}_{S,N}$

The vector from the point S to the point of action of the force is given by $\vec{\mathbf{r}}_{S,T} = -b\hat{\mathbf{i}}$ (Figure 17.17). Therefore the torque due to the force of the tendon $\vec{\mathbf{T}}$ on the ankle about the point S is then

$$\vec{\tau}_{S,T} = \vec{\mathbf{r}}_{S,T} \times \vec{\mathbf{T}} = -b\hat{\mathbf{i}} \times (T\cos\alpha\,\hat{\mathbf{i}} + T\sin\alpha\,\hat{\mathbf{j}}) = -bT\sin\alpha\,\hat{\mathbf{k}}\,.$$

(b) The torque diagram for the normal force is shown in Figure 17.18. The vector from the point S to the point where the normal force acts on the foot is given by

$\vec{\mathbf{r}}_{S,N} = (s\hat{\mathbf{i}} - h\hat{\mathbf{j}})$. Because the weight is evenly distributed on the two feet, the normal force on one foot is equal to half the weight, or $N = (1/2)mg$. The normal force is therefore given by $\vec{\mathbf{N}} = N\,\hat{\mathbf{j}} = (1/2)\,mg\,\hat{\mathbf{j}}$. Therefore the torque of the normal force about the point S is

$$\vec{\tau}_{S,N} = \vec{\mathbf{r}}_{S,N} \times N\,\hat{\mathbf{j}} = (s\hat{\mathbf{i}} - h\hat{\mathbf{j}}) \times N\,\hat{\mathbf{j}} = s\,N\,\hat{\mathbf{k}} = (1/2)\,s\,mg\,\hat{\mathbf{k}}\,.$$

(c) The force $\vec{\mathbf{F}}$ that the tibia exerts on the ankle will make no contribution to the torque about this point S since the tibia force acts at the point S and therefore the vector $\vec{\mathbf{r}}_{S,F} = \vec{\mathbf{0}}$.