

4.3 Velocity

When describing the motion of objects, words like “speed” and “velocity” are used in natural language; however when introducing a mathematical description of motion, we need to define these terms precisely. Our procedure will be to define average quantities for finite intervals of time and then examine what happens in the limit as the time interval becomes infinitesimally small. This will lead us to the mathematical concept that velocity at an instant in time is the derivative of the position with respect to time.

4.3.1 Average Velocity

The x -component of the **average velocity**, $v_{x,ave}$, for a time interval Δt is defined to be the displacement Δx divided by the time interval Δt ,

$$v_{x,ave} \equiv \frac{\Delta x}{\Delta t}. \quad (4.3.1)$$

Because we are describing one-dimensional motion we shall drop the subscript x and denote

$$v_{ave} = v_{x,ave}. \quad (4.3.2)$$

When we introduce two-dimensional motion we will distinguish the components of the velocity by subscripts. The average velocity vector is then

$$\bar{\mathbf{v}}_{ave} \equiv \frac{\Delta x}{\Delta t} \hat{\mathbf{i}} = v_{ave} \hat{\mathbf{i}}. \quad (4.3.3)$$

The SI units for average velocity are meters per second $[\text{m} \cdot \text{s}^{-1}]$. The average velocity is not necessarily equal to the distance in the time interval Δt traveled divided by the time interval Δt . For example, during a time interval, an object moves in the positive x -direction and then returns to its starting position, the displacement of the object is zero, but the distance traveled is non-zero.

4.3.3 Instantaneous Velocity

Consider a body moving in one direction. During the time interval $[t, t + \Delta t]$, the average velocity corresponds to the slope of the line connecting the points $(t, x(t))$ and $(t + \Delta t, x(t + \Delta t))$. The slope, the rise over the run, is the change in position divided by the change in time, and is given by

$$v_{ave} \equiv \frac{\text{rise}}{\text{run}} = \frac{\Delta x}{\Delta t} = \frac{x(t + \Delta t) - x(t)}{\Delta t}. \quad (4.3.4)$$

As $\Delta t \rightarrow 0$, the slope of the lines connecting the points $(t, x(t))$ and $(t + \Delta t, x(t + \Delta t))$, approach slope of the tangent line to the graph of the function $x(t)$ at the time t (Figure 4.4).

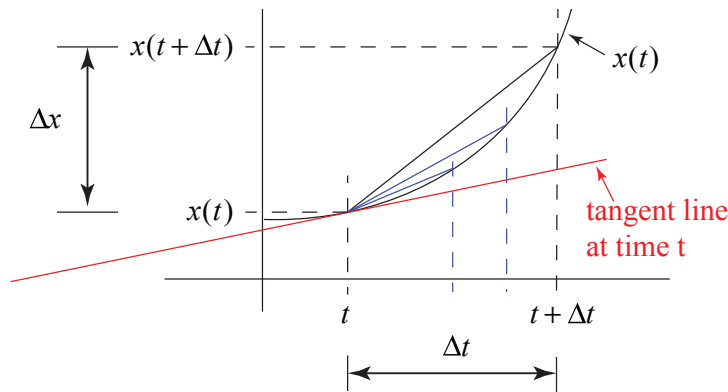


Figure 4.4 Plot of position vs. time showing the tangent line at time t .

The limiting value of this sequence is defined to be the x -component of the instantaneous velocity at the time t .

The x -component of *instantaneous velocity* at time t is given by the slope of the tangent line to the graph of the position function at time t :

$$v(t) \equiv \lim_{\Delta t \rightarrow 0} v_{ave} = \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} \equiv \frac{dx}{dt}. \quad (4.3.5)$$

The instantaneous velocity vector is then

$$\vec{v}(t) = v(t) \hat{\mathbf{i}}. \quad (4.3.6)$$

The component of the velocity, $v(t)$, can be positive, zero, or negative, depending on whether the object is travelling in the positive x -direction, instantaneously at rest, or the negative x -direction.

Example 4.1 Determining Velocity from Position

Consider an object that is moving along the x -coordinate axis with the position function given by

$$x(t) = x_0 + \frac{1}{2}bt^2 \quad (4.3.7)$$

where x_0 is the initial position of the object at $t = 0$. We can explicitly calculate the x -component of instantaneous velocity from Equation (4.3.5) by first calculating the displacement in the x -direction, $\Delta x = x(t + \Delta t) - x(t)$. We need to calculate the position at time $t + \Delta t$,

$$x(t + \Delta t) = x_0 + \frac{1}{2}b(t + \Delta t)^2 = x_0 + \frac{1}{2}b(t^2 + 2t\Delta t + \Delta t^2). \quad (4.3.8)$$

Then the x -component of instantaneous velocity is

$$v(t) = \lim_{\Delta t \rightarrow 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\left(x_0 + \frac{1}{2}b(t^2 + 2t\Delta t + \Delta t^2)\right) - \left(x_0 + \frac{1}{2}bt^2\right)}{\Delta t}. \quad (4.3.9)$$

This expression reduces to

$$v(t) = \lim_{\Delta t \rightarrow 0} \left(bt + \frac{1}{2}b\Delta t\right). \quad (4.3.10)$$

The first term is independent of the interval Δt and the second term vanishes because in the limit as $\Delta t \rightarrow 0$, the term $(1/2)b\Delta t \rightarrow 0$ is zero. Therefore the x -component of instantaneous velocity at time t is

$$v(t) = bt. \quad (4.3.11)$$

In Figure 4.5 we plot the instantaneous velocity, $v(t)$, as a function of time t .

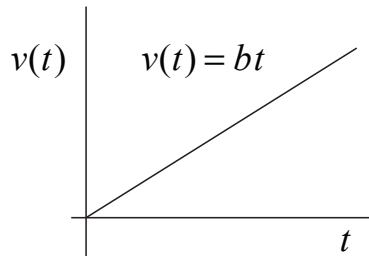


Figure 4.5 Plot of instantaneous velocity instantaneous velocity as a function of time.

Example 4.2 Mean Value Theorem

Consider an object that is moving along the x -coordinate axis with the position function given by

$$x(t) = x_0 + v_0 t + \frac{1}{2} b t^2. \quad (4.3.12)$$

The graph of $x(t)$ vs. t is shown in Figure 4.6.

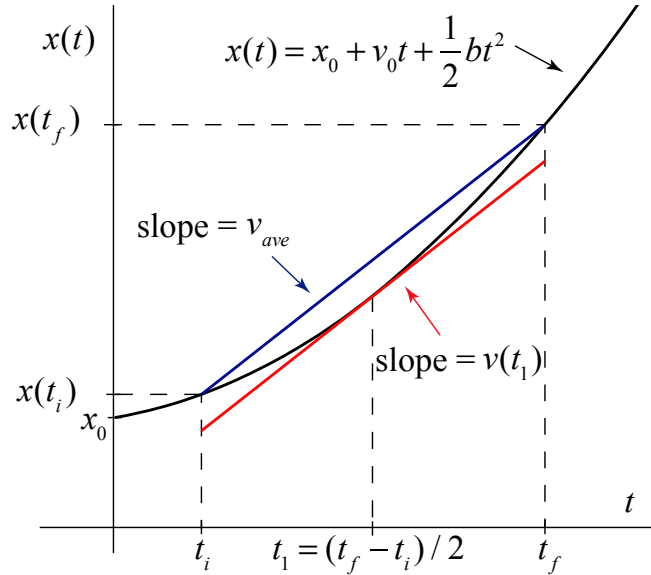


Figure 4.6 Intermediate Value Theorem

The x -component of the instantaneous velocity is

$$v(t) = \frac{dx(t)}{dt} = v_0 + b t. \quad (4.3.13)$$

For the time interval $[t_i, t_f]$, the displacement of the object is

$$x(t_f) - x(t_i) = \Delta x = v_0(t_f - t_i) + \frac{1}{2} b(t_f^2 - t_i^2) = v_0(t_f - t_i) + \frac{1}{2} b(t_f - t_i)(t_f + t_i). \quad (4.3.14)$$

Recall that the x -component of the average velocity is defined by the condition that

$$\Delta x = v_{ave}(t_f - t_i). \quad (4.3.15)$$

We can determine the average velocity by substituting Eq. (4.3.15) into Eq. (4.3.14) yielding

$$v_{ave} = v_0 + \frac{1}{2}b(t_f + t_i). \quad (4.3.16)$$

The Mean Value Theorem from calculus states that there exists an instant in time t_1 , with $t_i < t_1 < t_f$, such that the x -component of the instantaneously velocity, $v(t_1)$, satisfies

$$\Delta x = v(t_1)(t_f - t_i) . \quad (4.3.17)$$

Geometrically this means that the slope of the straight line (blue line in Figure 4.6) connecting the points $(t_i, x(t_i))$ to $(t_f, x(t_f))$ is equal to the slope of the tangent line (red line in Figure 4.6) to the graph of $x(t)$ vs. t at the point $(t_1, x(t_1))$ (Figure 4.6),

$$v(t_1) = v_{ave} . \quad (4.3.18)$$

We know from Eq. (4.3.13) that

$$v(t_1) = v_0 + bt_1 . \quad (4.3.19)$$

We can solve for the time t_1 by substituting Eqs. (4.3.19) and (4.3.16) into Eq. (4.3.18) yielding

$$t_1 = (t_f + t_i) / 2 \quad (4.3.20)$$

This intermediate value $v(t_1)$ is also equal to one-half the sum of the initial velocity and final velocity

$$v(t_1) = \frac{v(t_i) + v(t_f)}{2} = \frac{(v_0 + bt_i) + (v_0 + bt_f)}{2} = v_0 + \frac{1}{2}b(t_f + t_i) = v_0 + bt_1 . \quad (4.3.21)$$

For any time interval, the quantity $(v(t_i) + v(t_f)) / 2$, is the arithmetic mean of the initial velocity and the final velocity (but unfortunately is also sometimes referred to as the average velocity). The average velocity, which we defined as $v_{ave} = (x_f - x_i) / \Delta t$, and the arithmetic mean, $(v(t_i) + v(t_f)) / 2$, are only equal in the special case when the velocity is a linear function in the variable t as in this example, (Eq. (4.3.13)). We shall only use the term average velocity to mean displacement divided by the time interval.

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8.01 Classical Mechanics
Fall 2016

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