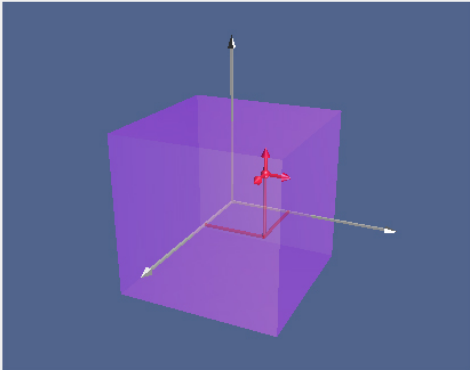# Problem Solving 1: The Mathematics of 8.02

Part I. Coordinate Systems

In 8.02 we regularly use three different coordinate systems: rectangular (Cartesian), cylindrical and spherical. In order to become familiar with the unit vectors for each of those systems, take a minute to explore the online visualizations . Go to the *Vector Fields* page and select the *Coordinate Systems* Shockwave (next to bottom column). Once it loads you will see the following page:

Part II. Differentiation & The Gradient

For one dimensional functions of a single variable, e.g. $f(x)$, the derivative, $\frac{df}{dx}$ or $f'(x)$, tells you how the function changes for small changes in the independent variable (x). The picture of this as the slope of the function is helpful. When a function depends on several variables, e.g. $f(x, y, z)$, there are several different derivatives, called partial derivates, $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ & $\frac{\partial f}{\partial z}$. Both in concept and in practice the partial derivative is the same as the one dimensional derivative, asking how the function changes as you change one of its independent variables (while holding the others fixed – treat them as constants when you take the derivative).

PROBLEM 2: (*answer on the tear-sheet at the end*)**:**

Consider the function $V(x, y, z) = \dfrac{kq}{r} = \dfrac{kq}{\sqrt{x^2 + y^2 + z^2}}$. Calculate $\dfrac{\partial V}{\partial x}, \dfrac{\partial V}{\partial y}$ & $\dfrac{\partial V}{\partial z}$.

It is also convenient to define the gradient of a multidimensional scalar function. The gradient is a vector that everywhere points in the direction of steepest ascent (uphill) of the function. It is calculated as follows: $\text{gradient}(f) = \vec{\nabla} f = \dfrac{\partial f}{\partial x}\hat{\mathbf{i}} + \dfrac{\partial f}{\partial y}\hat{\mathbf{j}} + \dfrac{\partial f}{\partial z}\hat{\mathbf{k}}$.

PROBLEM 3: (*answer on the tear-sheet at the end*)**:**

Again consider $V(x, y, z) = \dfrac{kq}{r} = \dfrac{kq}{\sqrt{x^2 + y^2 + z^2}}$. Calculate its gradient $\vec{\nabla} V$.

PROBLEM 3: *continued* (*answer on the tear-sheet at the end*):

Rewrite the gradient as a function of $r = \sqrt{x^2 + y^2 + z^2}$ and $\hat{r} = \dfrac{\vec{\mathbf{r}}}{r} = \dfrac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{r}$

Does $\vec{\nabla}V$ always point "uphill" for the function *V*? Explain.

Part III. Multivariable Integration

Although we frequently use integration in 8.02, the integrals we do tend to be VERY easy (most things are constant and pull straight out of the integral). Where students typically have problems is in not understanding the concept of integration and how to set up the integral. Below we focus on the types of integrals used in this class. If you find yourself doing "a lot of math" then you probably aren't doing it right. We leave the math intensive problems to 18.02.

A. Concept: Integration as Summing the Pieces

In single variable calculus the definite integral of a function is often written: $\displaystyle\int_a^b f(x)dx$.

The *dx* is a differential – a small length of the x axis – and to many seems to be merely a notational requirement, like an appendage of the integral. I encourage you to think of the differential as the most important part of the integral, and to start every integral with ONLY a differential. For example, if you want to know the total charge Q on some strangely shaped object with a given charge density, you would want to write: $Q = \displaystyle\int dQ$. If you want to know the area of some surface you would start $A = \displaystyle\iint dA$.

What have we achieved by doing this, you might ask? We have decided that in order to find out the total value of something (the charge or the area in the two examples above) we need to break it up into parts and add up the contributions from each of those parts. The differential (e.g. *dQ* or *dA*) is some very small piece of the whole that (1) we can easily write down and (2) provides us some way to move through all the pieces of that whole.

Example 1

Suppose we want to find the area under a function $f(x)$ between $x = a$ and $x = b$. To do this we break the total area A into a number of small rectangles of width dx and height $f(x)$. The area of each of these little rectangles dA is easily written (its just the area of a rectangle):

$$dA = f(x)\ dx.$$

To calculate the area we write: $A = \iint dA = \int_a^b f(x)dx$
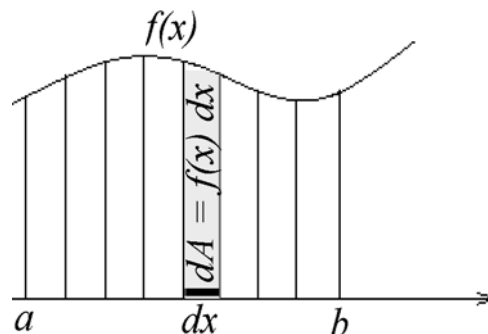
Our choice of dA obeyed our two rules: it was easy to write down (we know the area of rectangles) and by adding up the area of such rectangles as we walk x from a to b we will get the value of the whole area.

B. Notation: Multivariable Integration

When we move to integrals of multivariable functions we need some new notation. Instead of only having one straight axis we can walk along (i.e. the x axis in the above example), in three dimensions we can walk along curves (one dimensional objects), surfaces (2D) or volumes (3D). For each new dimension we need a new variable to tell us where we are on the object – a single variable, distance from the end, tells you where you are on a curve, but two variables are needed to tell you where you are on a surface – and hence an extra integral sign. So we have:

Line (Path) Integrals:	$s = \int ds$
Surface Integrals:	$A = \iint dA$
Volume Integrals:	$V = \iiint dV$

For line and surface integrals we have one more notational convention. We distinguish between "open" and "closed" curves and surfaces. Closed curves are those where the beginning and end points are the same (the perimeter of a circle is closed, a line isn't). Closed surfaces are those that completely contain volume (a spherical shell is closed, a plane isn't). We indicate that we are integrating over a closed curve or surface by putting a circle over the integral sign:

Closed Line Integral: $s = \oint ds$ Closed Surface Integral: $A = \oiint dA$

This distinction is irrelevant for *doing* the integral, but helps in remembering the shape over which you are integrating.

C. Differentials in different coordinate systems

A final piece of useful information is how to break an object up into small pieces in different coordinate systems. In rectangular (Cartesian) coordinates, the geometry is very straightforward. Distances (for line integrals) tend to have $ds = dx$ (or dy or dz). Areas are just squares ($dA = dx\ dy$ or $dx\ dz$ …). Volumes are just cubes ($dV = dx\ dy\ dz$).

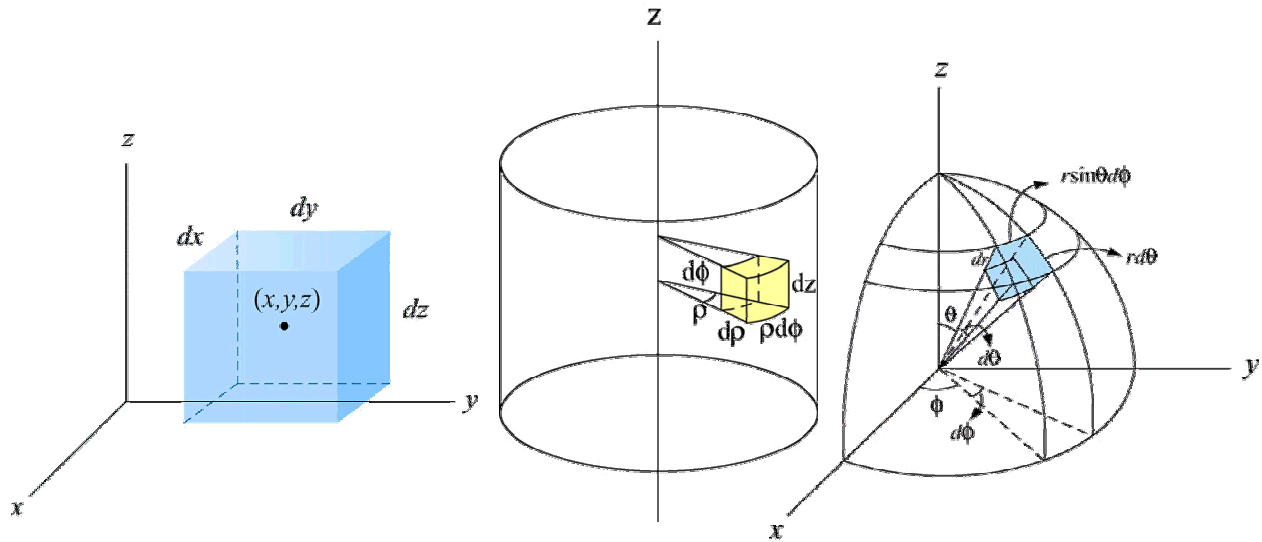Pictures are helpful for finding similar quantities in cylindrical and spherical coordinates:

Figure 2: Differential dimensions in rectangular, cylindrical & spherical coordinate systems.
Cylindrical coordinates (ρ, ϕ, z) has differential "cube" of sides $d\rho$, $\rho d\phi$ and dz
Spherical coordinates (r, θ, ϕ) has differential "cube" of sides dr, $rd\theta$ and $r\sin\theta\ d\phi$

In addition to integrating over these small differential cubes, it is often convenient to integrate over larger differential objects. For example, in example 1 we wrote $dA = f(x)\ dx$ and integrated only in x, rather than writing $dA = dx\ dy$ and integrating from $y = 0$ to $f(x)$ and then over x. We essentially did the *dy* integral in our head, because it was straightforward.

Example 2

Write integral expressions for the area of a circle of radius R in two different ways
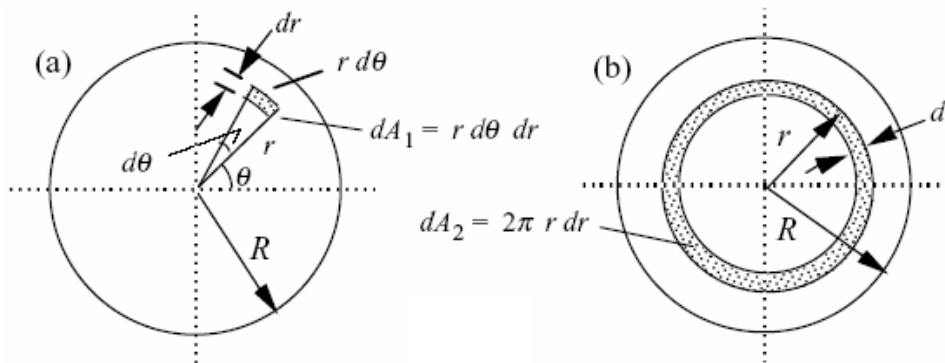
(a) $A = \displaystyle\iint_{Circle} dA = \int_{r=0}^{R}\int_{\theta=0}^{2\pi} (rd\theta)dr$ \qquad\qquad (b) $A = \displaystyle\iint_{Circle} dA = \int_{r=0}^{R} 2\pi r\,dr$

PROBLEM 4: (*answer on the tear-sheet at the end*)**:**

Try this yourself in cylindrical coordinates. Write down integral expressions for the volume of a cylinder of radius *R* and height *H* as

(a) A 3D integral, integrating over the small differential "cubes" shown in figure 2

(b) A 2D integral, integrating over circles of radius *r*, thickness *dr* and height *dz*

(c) A 1D integral, integrating over cylindrical shells of radius *r*, thickness *dr* and height *H*

(d) A 1D integral, integrating over disks of radius *R*, thickness d*z*

D. Charge Density: An Example of Integrating Scalar Functions

Instead of being confined to a point, charge is often distributed over objects, either uniformly (with constant charge density) or with some position dependent charge density. For 1- 2- and 3-dimensional objects we us different symbols for charge density:

1D: $dq = \lambda\, ds$ (λ in [C/m])   2D: $dq = \sigma\, dA$ (σ in [C/m^2])   3D: $dq = \rho\, dV$ (ρ in [C/m^3])

PROBLEM 5: (*answer on the tear-sheet at the end*)**:**

Calculate the total charge on each of the following objects, given their charge density

(a) A ring, radius R, with constant linear charge density λ

PROBLEM 5 *continued*: (*answer on the tear-sheet at the end*):

(b) A solid sphere, radius R, with volume charge density $\rho(r) = \rho_R R / r$

E. Line Integrals of Vector Functions dotted into Displacement

In addition to integrating scalar functions, we often integrate vector functions which are dotted into some distance. This dot product is then a scalar function, and we integrate just as above. For example, if an object moves a distance $d\vec{s}$ while a force $\vec{F}$ acts on it, we say that the force does work $dW = \vec{F} \cdot d\vec{s}$. If the object then moves along a path, we can calculate the total work done by the force while the object was moving as:

$$W = \int dW = \int \vec{F} \cdot d\vec{s}$$

The dot product indicates that only the part of the force pushing along the displacement contributes to the total work done.

PROBLEM 6: (*answer on the tear-sheet at the end*):

A charge q in a uniform electric field $\vec{E} = E_0 \hat{\mathbf{j}}$ feels a force $\vec{F} = q\vec{E}$. How much work is done by the field on the particle if it moves along a semi-circle of radius R centered at the origin in the xy plane from $(x, y) = (0, R)$ to $(0, -R)$?

D. Surface Integrals of Vector Functions dotted into Normals: Flux

In two dimensions, the equivalent of the question we just asked in 1D – "how much does this vector field point *along* the path we are integrating along?" – is "how much does this vector field push *through* the surface we are integrating over?" This quantity is called the flux, and for a vector function $\vec{F}$, the flux through a surface S is given by

$$\iint_S \vec{F} \cdot d\vec{A} = \iint_S \vec{F} \cdot \hat{n}\, dA = \iint_S F_n\, dA$$

where $d\vec{A} = dA\,\hat{n}$ and $\hat{n}$ is a unit vector pointing normal (perpendicular) to the surface. The dot product $F_n = \vec{F} \cdot \hat{n}$ is the component of $\vec{F}$ parallel to $\hat{n}$.

As an example, suppose the function $\vec{F}$ describes the rate at which water flows through a cylindrical pipe per cross sectional area per unit time (i.e. has units liter/m^2 s). The *flux* of that function through some area A is the rate at which water hits that area. Intuitively, that rate should be independent of the specific area, as long as it completely covers the pipe.

Example 3

Show that the flux of a uniform flow $\vec{F} = f_0\,\hat{k}$ through a pipe of radius R is the same through a flat disk and through a hemisphere (both of which completely fill the cross-section of the pipe).

From the definition of the flow, we can see that the pipe runs along the z-axis. A flat disk will have a normal parallel to the flow ($\hat{n} = \hat{k}$) and the integral is straight forward:

$$\text{Flux} = \iint_S \vec{F} \cdot d\vec{A} = \iint_S f_0\, dA = f_0 \pi R^2$$

The hemisphere is a little trickier because its normal vector is in the $\hat{r}$ direction, so

$$\text{Flux} = \iint_S \vec{F} \cdot d\vec{A} = \iint_S f_0\,\hat{k} \cdot \hat{r}\, dA = \iint_S f_0\, \cos(\theta)\, dA = \int_{\theta=0}^{\pi/2} \int_{\varphi=0}^{2\pi} f_0\, \cos(\theta)\{R\sin(\theta)\,d\varphi\}\{R\,d\theta\}$$

$$= f_0 \pi R^2 \int_{\theta=0}^{\pi/2} 2\sin(\theta)\cos(\theta)\,d\theta = f_0 \pi R^2 \sin^2(\theta)\Big|_{\theta=0}^{\pi/2} = f_0 \pi R^2$$

It's the same for the disk and the hemisphere, just as it should be – the same amount of water must pass through both of them because they both completely cover the cross-section of the pipe, so all water that flows through one must flow through the other.

PROBLEM 7: *(Answer on the tear-sheet at the end!)*

(a) Consider a uniform electric field $\vec{\mathbf{E}} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}}$ which intersects a surface of area A. What is the electric flux through this area if the surface lies (i) in the xz plane with normal in the positive y direction? (ii) in the xy plane with the normal in the positive z direction?

(b) A cylinder has radius R and height h, oriented along the z-axis. A uniform field $\vec{\mathbf{E}} = E_o\hat{\mathbf{j}}$ penetrates the cylinder. Determine the flux $\iint_S \vec{\mathbf{E}} \cdot \hat{\mathbf{n}}\, dA$ for the side of the cylinder with $y > 0$, where the area normal points away from the interior of the cylinder.

Hints: If ϕ is the angle in the xy plane measured from the x-axis toward the positive y-axis, what is the vector formula for the normal $\hat{\mathbf{n}}$ to the side of the cylinder with $y > 0$, in terms of ϕ, $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$?

What is $\vec{\mathbf{E}} \cdot \hat{\mathbf{n}}$? What is the differential area of the side of the cylinder in term of R, dz, and $d\phi$?
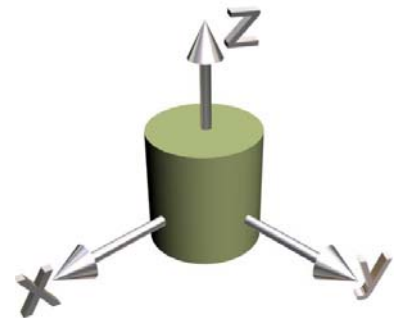
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

Tear off this page and turn it in at the end of class !!!!
Note: Writing in the name of a student who is not present is a Committee on
Discipline offense.

Names _____

_____

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PROBLEM 1:
What is it about the coordinate axes in cylindrical and spherical coordinates that makes those
axes very different from the axes in a Cartesian coordinate system?

PROBLEM 2:

(a) $\dfrac{\partial V}{\partial x} =$

(b) $\dfrac{\partial V}{\partial y} =$

(c) $\dfrac{\partial V}{\partial z} =$

PROBLEM 3:

(a) $\vec{\nabla} V =$

(b) Rewritten in terms of r and $\hat{\mathbf{r}}$, $\vec{\nabla} V(r) =$

(c) Does $\vec{\nabla} V$ always point "uphill" for the function V? Explain.

PROBLEM 4:

Write down expressions for the volume of a cylinder of radius R and height H as

(a) A 3D integral, integrating over the small differential "cubes" shown in figure 2

(b) A 2D integral, integrating over circles of radius r, thickness dr and height dz

(c) A 1D integral, integrating over cylindrical shells of radius r, thickness dr and height H

(d) A 1D integral, integrating over disks of radius R, thickness dz

PROBLEM 5:

Calculate the total charge on each of the following objects, given their charge density

(a) A ring, radius R, with constant charge density λ $\qquad\qquad$ Q =

(b) A solid sphere, radius R, with charge density $\rho(r) = \rho_R R/r$ $\qquad$ Q =

PROBLEM 6:

How much work is done by a uniform electric field $\vec{E} = E_0 \hat{\mathbf{j}}$ of a charged particle q that moves along a semi-circle of radius R centered at the origin in the xy plane from $(x, y) = (0, R)$ to $(0, -R)$?

$\qquad$ **W =**

PROBLEM 7:

(a) Consider a uniform field $\vec{E} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}}$ which intersects a surface of area A. What is the flux through this area if the surface lies

(i) in the xz plane?

(ii) in the xy plane?

(b) Determine the flux $\iint_S \vec{E} \cdot \hat{\mathbf{n}} \, dA$ for the side of the cylinder with $y > 0$.