MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

Problem Solving 9: Driven LRC Circuits

OBJECTIVES

1. To explore the relationship between driven current and driving *emf* in three simple circuits that contain: (1) only resistance; (2) only inductance; and (3) only capacitance.

2. To examine these same relationships in the general case where *R*, *L*, and *C* are all present, and to do two sample problems on the *LRC* circuits.

REFERENCE: Sections 12.1 – 12.4, 8.02 Course Notes.

General Properties of Driven LRC Circuits

An LRC circuit is the electrical analog of a mass on a spring. We distinguish two behaviors. In the first, we consider its "free" oscillations that occur when we "kick" the circuit (charge the capacitor or send a constant current through) and then stand back and watch it oscillate. If we do this we will see a natural frequency of oscillation that decays in a finite time.

A second behavior emerges if we "drive" the *LRC* circuit with a source of *emf* with some (arbitrary) amplitude and frequency. If we drive the circuit with an *emf* $V(t) = V_0 \sin \omega t$, where ω is any frequency we desire (we get to pick this) and V_0 is any amplitude we desire, then the "driven" response of the system is given by

$$I(t) = I_0 \sin(\omega t - \phi)$$

where

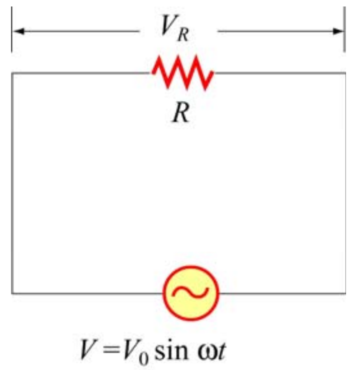
$$I_0 = \frac{V_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}, \quad \tan\phi = \left(\frac{\omega L - \frac{1}{\omega C}}{R}\right) \qquad (8.1)$$

Note the "driven" response is at the (arbitrary) frequency of the driver, and ***not*** at the natural frequency of the system. However the system will show maximum response to the driving *emf* when the driving frequency ***is*** at the natural frequency of oscillation of the system, i.e. when $\omega = 1/\sqrt{LC}$. We can compute the average power consumed by the circuit by calculating the time average of $I(t)V(t)$ (see Section 12.4, 8.02 Course Notes):

$$\langle P(t) \rangle = \langle I(t)V(t) \rangle = \frac{1}{2} I_0 V_0 \cos\phi \qquad (8.2)$$

Example 1: Driven circuit with resistance only

We begin with a circuit which contains only resistance. The circuit diagram is shown below.



The circuit equation is

$$I_R(t)R - V(t) = 0 .$$

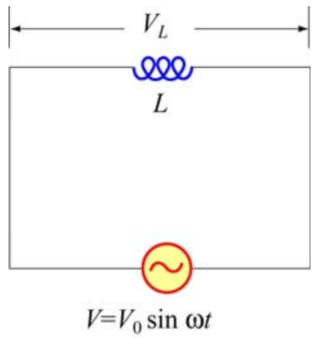
Question 1: What is the amplitude I_{R0} and phase ϕ of the current $I_R(t) = I_{R0}\sin(\omega t - \phi)$?
Answer: (*answer this and subsequent questions on the tear-off sheet at the end*)

Question 2: What values of L and C do you choose in the general equation (8.1) to reproduce the result you obtained in your answer above? HINT: This is as easy (and as strange) as you probably first think.
Answer:

Question 3: What is the *time-averaged* power $\langle P_R(t)\rangle = \langle I_R(t)V_R(t)\rangle$ dissipated in this circuit? You will need to know that the time average of $\sin^2 \omega t$ is $\langle \sin^2 \omega t \rangle = 1/2$.
Answer:

Example 2: Driven circuit with inductance only

Now suppose the voltage source $V(t) = V_0 \sin(\omega t)$ is connected in a circuit with only self-inductance. The circuit diagram is



The circuit equation is

$$V(t) = L\frac{dI}{dt}$$

Question 4: Solve the above equation for the current as a function of time. If we write this current in the form $I_L(t) = I_{L0}\sin(\omega t - \phi)$, what is the amplitude I_{L0} and phase ϕ of the current? You may need to use the trigonometric identity that $\sin(\omega t - \phi) = \sin\omega t \cos\phi - \sin\phi\cos\omega t$.

Answer:

Question 5: What values of R and C do you choose in the general equation (8.1) to reproduce the result you obtained in the question above?

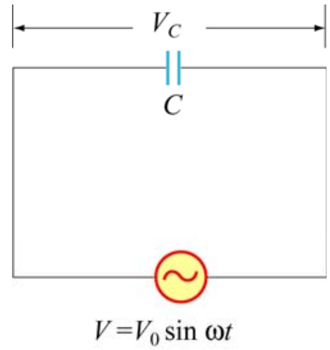
Answer:

Question 6: What is the *time-averaged* power $\langle P_L(t)\rangle = \langle I_L(t)V_L(t)\rangle$ dissipated in this circuit? You will need to know that the time-average of $\sin\omega t\cos\omega t$ is $\langle \sin\omega t\cos\omega t\rangle = 0$.

Answer:

Example 3: Driven circuit with capacitance only

The ac voltage source $V(t) = V_0 \sin(\omega t)$ is connected in a circuit with capacitance only. The circuit diagram is



The circuit equation for this circuit is

$$\frac{Q}{C} - V(t) = 0$$

If we take the time derivative of this equation we get

$$\frac{I_C}{C} - \frac{d}{dt}V(t) = \frac{I_C}{C} - \omega V_0 \cos \omega t = 0$$

Question 7: Solve the above equation for the current as a function of time. If we write this current in the form $I_C(t) = I_{C0} \sin(\omega t - \phi)$, what is the amplitude I_{C0} and phase ϕ of the current? You may need to use the trigonometric identity that $\sin(\omega t - \phi) = \sin \omega t \cos \phi - \sin \phi \cos \omega t$.

Answer:

Question 8: What is the *time-averaged* power $\langle P_C(t) \rangle = \langle I_C(t) V_C(t) \rangle$ dissipated in this circuit? You will need to know that the time-average of $\sin \omega t \cos \omega t$ is $\langle \sin \omega t \cos \omega t \rangle = 0$.

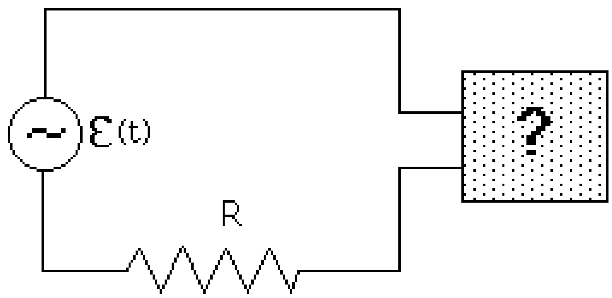
Answer:

Sample Problem 1

The circuit shown below contains an AC generator which provides a source of sinusoidally varying emf $\varepsilon(t) = \varepsilon_0 \sin\omega t$, a resistor with resistance R, and a "black box", which contains ***either*** an inductor ***or*** a capacitor, ***but not both***. The amplitude of the driving emf, ε_0, is $100\sqrt{2}$ Volts, and the angular frequency ω is 10 rad/sec. We measure the current in the circuit and find that it is given as a function of time by the expression: $I(t)$ = (10 Amps) $\sin(\omega t + \pi/4)$ [Note: $\pi/4$ radians = 45°, $\tan(\pi/4) = +1$].

Question 9: Does this current lead or lag the emf $\varepsilon(t)=\varepsilon_0 \sin(\omega t)$

Answer:



Question 10: What is the unknown circuit element in the black box--an inductor or a capacitor?

Answer:

Question 11: What is the numerical value of the resistance R? Indicate units.

Answer:

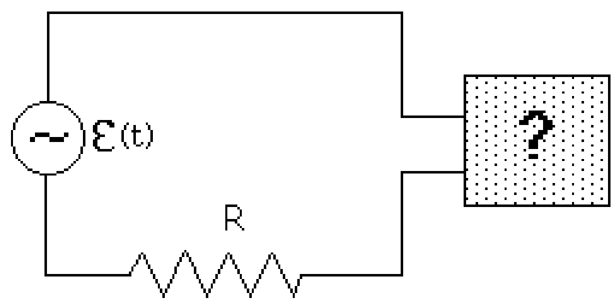
Question 12: What is the numerical value of the capacitance *or* of the inductance, as the case may be?. Indicate units.

Answer:

Sample Problem 2

The circuit shown below contains an AC generator which provides a source of sinusoidally varying emf $\varepsilon(t) = \varepsilon_0 \sin(\omega t)$, a resistor with resistance $R = 3\ \Omega$, and a "black box", which contains ***either*** an inductor ***or*** a capacitor, ***or both***. The amplitude of the driving emf, ε_0, is 6 Volt. We measure the current in the circuit at an angular frequency $\omega = 1$ radians/sec and find that it is exactly in phase with the driving emf. We measure the current in the circuit at an angular frequency $\omega = 2$ radians/sec and find that it is out of phase with the driving emf by exactly $\pi/4$ radians.

Question 13: What does the black box contain – an inductor or a capacitor, or both? Explain your reasoning. Does current lead or lag at $\omega = 2$ radians/sec?
Answer:

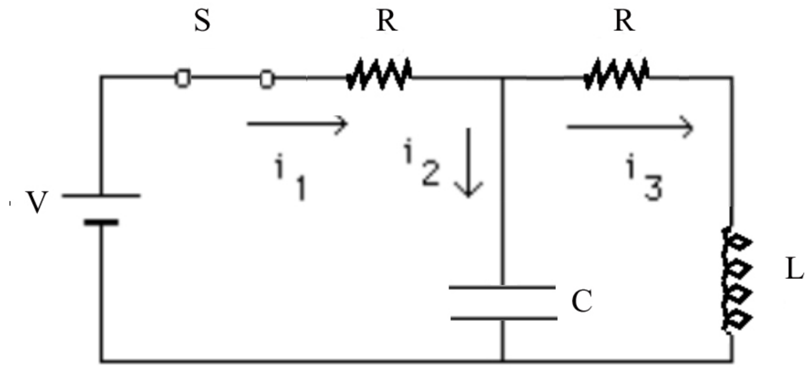


Question 14: What is the numerical value of the capacitance *or* of the inductance, *or of both*, as the case may be? Indicate units. Your answer(s) will involve simple fractions only, you will not need a calculator to find the value(s).
Answer:

Question 15: What is numerical value of the *time-averaged* power dissipated in this circuit when $\omega = 1$ radians/sec? Indicate units, that is the time-average of $I(t)V(t)$. You will need to know that the time-average of $\sin^2 \omega t$ is ½.
Answer:

Sample Exam Question (If time, try to do this by yourself, closed notes)

A circuit consists of a battery with *emf V*, an inductor *L*, a capacitor *C*, and two resistors, each with resistance *R*, as shown in the sketch. The capacitor is initially uncharged and there is no current flowing anywhere in the circuit. The switch S has been open for a long time, and is then closed, as shown in the diagram.



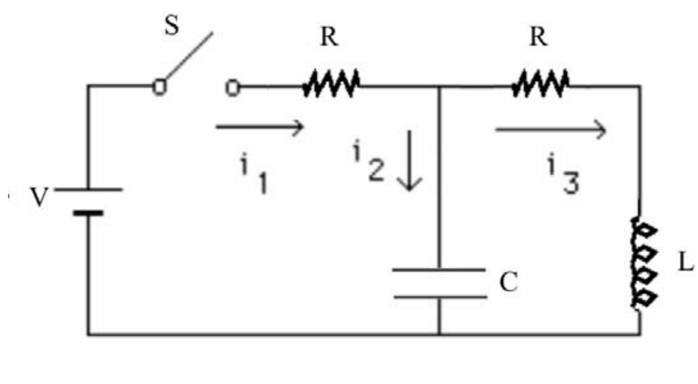
(a) Using Kirchhoff's Loop Rule as modified for inductors, what is the sum of the potential drops around the outer loop (the loop including both the emf and the inductor) if we move clockwise around the loop?

(b) ***Just after*** the switch *S* is closed, what are the currents i_1, i_2, and i_3 in terms of the given quantities? ***Assume that the left loop of the circuit has zero inductance.*** You do ***not*** have to solve any differential equations to answer this question.

(c) A long, long time after switch *S* is closed, what are the currents i_1, i_2, and i_3? You do ***not*** have to solve any differential equations to answer this question.

(d) A long, long time after switch *S* is closed, what is the charge on the capacitor? You do ***not*** have to solve any differential equations to answer this question.

(e) The switch S is now opened again. ***Just after*** the switch is opened again, what are the currents i_1, i_2, and i_3 in terms of the given quantities***? Assume that the left loop of the circuit has zero inductance.***



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Tear off this page and turn it in at the end of class !!!!

Note: Writing in the name of a student who is not present is a COD offense.

Problem Solving 10: Driven RLC Circuits

Group ____________________________________ (**e.g. L02 6A Please Fill Out)**

Names ____________________________________

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Example 1: Driven circuit with resistance only

Question 1: What is the amplitude I_{R0} and phase ϕ of the current $I_R(t) = I_{R0}\sin(\omega t - \phi)$?

Answer: I_{R0}:________________________ ϕ:_________________

Question 2: What values of L and C do you choose in the general equation (8.1) to reproduce the result you obtained in your answer above?

Answer: L:________________________ C:_________________

Question 3: What is the *time-averaged* power $\langle P_R(t)\rangle = \langle I_R(t)V_R(t)\rangle$ dissipated?

Answer: $\langle P_R(t)\rangle =$ __________________________

Example 2: Driven circuit with inductance only

Question 4: What is the amplitude I_{L0} and phase ϕ of the current $I_L(t) = I_{L0}\sin(\omega t - \phi)$?

Answer: I_{L0}:__________________________ ϕ:___________________

Question 5: What values of R and C do you choose in the general equation (8.1) to reproduce the result you obtained in the question above?

Answer: R: ________________________ C:_________________

Question 6: What is the *time-averaged* power $\langle P_L(t)\rangle = \langle I_L(t)V_L(t)\rangle$ dissipated?

Answer: $\langle P_L(t)\rangle =$ ___________________________

Example 3: Driven circuit with capacitance only

Question 7: What is the amplitude I_{C0} and phase ϕ of the current $I_C(t) = I_{C0}\sin(\omega t - \phi)$?

Answer: I_{C0} :_________________________ ϕ :__________________

Question 8: What is the *time-averaged* power $\langle P_C(t)\rangle = \langle I_C(t)V_C(t)\rangle$ dissipated?

Answer: $\langle P_C(t)\rangle =$ ___________________________

Sample Problem 1:
Question 9: Does this current lead or lag the emf $\varepsilon(t) = \varepsilon_0 \sin\omega t$

Answer: _______________________

Question 10: What is the unknown circuit element in the black box--an inductor or a capacitor?
Answer: _______________________

Question 11: What is the numerical value of the resistance *R*? Indicate units.

Answer: ____________________________

Question 12: What is the numerical value of the capacitance *or* of the inductance? Indicate units.

Answer: _______________________

Sample Problem 2:

Question 13: What does the black box contain--an inductor or a capacitor, or both? Explain your reasoning. Does the current lead or lag at $\omega = 2$ radians/sec?
Answer:

Question 14: What is the numerical value of the capacitance *or* of the inductance, *or of both*, as the case may be? Indicate units. Your answer(s) will involve simple fractions only, you will not need a calculator to find the value(s).

Answer: *L:* ____________________ *C:* _____________________

Question 15: What is numerical value of the *time-averaged* power dissipated in this circuit when ω =1 radians/sec? Indicate units.

Answer: _________________________________