8.022 (E&M) – Lecture 11

Topics:

- Introduction to Special Relativity
  - Length contraction and Time dilation
  - Lorentz transformations
  - Velocity transformation

Special relativity

- Ready for the challenge?
  - Special relativity seems easy but it's not!
  - A new way of thinking that often goes against intuition
  - It will take some time to "digest it", but believe me: it's worth the effort!
- Why do we need it in 8.022?
  - Weren't you frustrated last time when magnetic forces came out of nowhere?
  - Special relativity naturally explains them in terms of electric forces seen from in a reference frame in motion
- This is important for everybody
  - Physics majors: first of many iterations on a crucial topic
  - Non Physics majors: chance to know what you are missing
    - Don't forget: you are still in time…

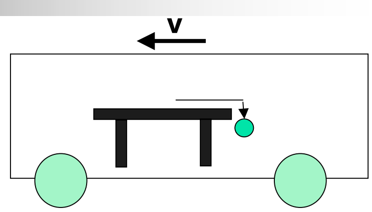
The principles of special relativity

- Formulated in 1905 by A. Einstein
  - Incredible but true:

    no Nobel Prize for this!
- Based upon 2 postulates
  - The laws of physics are the same for all reference frames
  - The speed of light is the same (c) in all reference frames
- (Inertial) Reference frame
  - System of coordinates in which the observer is non accelerating (inertial = non accelerating)

Reference frames: examples

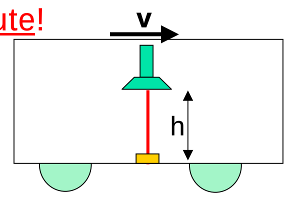
- Situation
  - A train is moving with velocity **v** w.r.t. to a station
  - A table is anchored to the train
  - A ball is falling from the table



- We can identify 3 systems of reference and 3 observers:
  - Observer 1: sitting on a bench at the station
  - Observer 2: sitting on the table on the train
  - Observer 3: a bug sitting on top of the falling ball
- Who are the observers in an inertial reference frame?
  - Observers 1 and 2
  - Observer 3 is not: the ball is falling with acceleration g

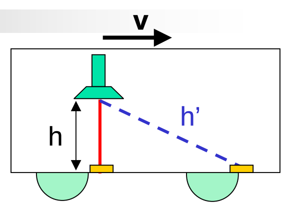
Is time the same in all reference frames?

- These (apparently) innocent assumptions have amazing consequences such as <u>time is not absolute</u>!
- Problem
  - The train is moving with velocity **v** // x axis
    - Observer 1: standing in the train
    - Observer 2: at the station
  - Observer 1 flashes a pulse of light vertically to a photosensor mounted on the floor of the train
  - Both observers measure the time between when the light is emitted and when the light reaches the sensor



Will the 2 observers measure the same time?

Time in different reference frames



- Let's calculate time measured by the 2 observers
- Train reference frame (observer 1)

$$\begin{cases}\text{Distance traveled by light: h} \\ \text{Velocity of light: c}\end{cases} \Rightarrow \Delta t = \Delta t_1 = \frac{h}{c}$$

- Station reference frame (observer 2)

$$\begin{cases}\text{Distance traveled by light: } h' = \sqrt{h^2 + (v\Delta t_2)^2} \\ \text{Velocity of light: c}\end{cases} \quad \Delta t' = \Delta t_2 = \frac{h'}{c}$$

$$(\Delta t_2)^2 = \left(\frac{h'}{c}\right)^2 = \frac{h^2 + (v\Delta t_2)^2}{c^2} = \Delta t_2^{\,2} + \frac{v^2}{c^2}\Delta t_2^{\,2} \quad \Rightarrow \quad \Delta t_1 = \Delta t_2\sqrt{1 - \frac{v^2}{c^2}}$$

Defining $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$ $\quad \Rightarrow \quad$ $\boxed{\Delta t' = \gamma\ \Delta t}$

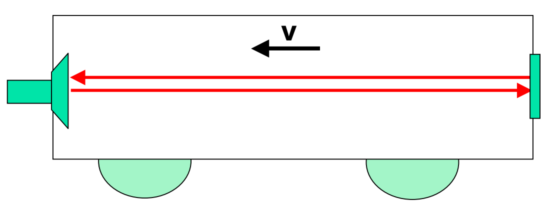
Time dilation

- We just derived a very important result!
- Gamma factor: $\gamma \equiv \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\beta^2}} > 1 \quad \text{with } \beta \equiv \beta_v \equiv \frac{v}{c}$
- Since $\Delta t' = \gamma \Delta t$ ➔ $\Delta t'$ is always larger than Δt
  - $\Delta t'$ = time measured by the observer in the station who sees the clock in motion
  - Δt = time measured by the observer on the train, at rest wrt the clock
- Conclusion:

  Clocks in motion run slower (time dilation) $\Delta t' = \gamma\ \Delta t$

Length in different reference frames

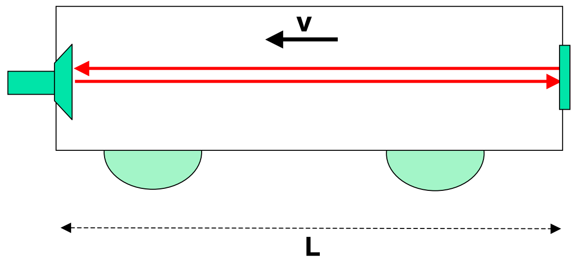
- Problem 2
  - Now observer 1 flashes a pulse of light horizontally from left end of the train
  - The light is reflected by a mirror on the right end wall and detected by a photosensor on the left wall



What is the length of the train measured by each observer?

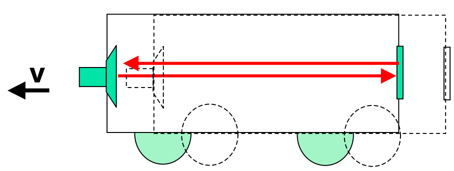
Length in train reference frames

- For observer in train reference frame
  - Events we are interested in: emission and reception of light
    - Time in between the two: $\Delta t = \Delta t_{train}$
    - Length of the train: $L = \frac{c \Delta t}{2} \Rightarrow \Delta t = \frac{2L}{c}$



Length in the station reference frame

- Calculate separately Δx_1 (L→R) and Δx_2 (R→L)



$$\Delta t'_1 = (L' - v\Delta t'_1)/c$$
$$\Delta t'_2 = (L' + v\Delta t'_2)/c$$

  - Δt_1 is shorter because train and light move in opposite directions
  - Δt_2 is longer because train and light move in the same direction
  - L' (t') = length (time) measured from station reference frame

- Rearrange terms: $\begin{cases} \Delta t'_1 = \frac{L'}{c+v} \\ \Delta t'_2 = \frac{L'}{c-v} \end{cases}$

Length contraction

- Total time in the station reference frame = sum of $\Delta t'_1$ and $\Delta t'_2$:

$$\Delta t' = \Delta t'_1 + \Delta t'_2 = \frac{L'}{c-v} + \frac{L'}{c+v} =$$

$$= L'\frac{2c}{c^2 - v^2} = L'\frac{2c}{c^2(1-\frac{v^2}{c^2})} = \frac{2L'\gamma^2}{c}$$

- Remember how time dilates: $\Delta t' = \gamma \Delta t$ →

$$\frac{2L'\gamma^2}{c} = \Delta t' = \gamma \Delta t = \gamma \frac{2L}{c} \quad \Rightarrow \quad \boxed{L' = \frac{L}{\gamma}}$$

- Since $\gamma > 1$ →
  Moving objects appear contracted (length contraction)

Summary so far

- Assume Special Relativity postulates hold:
  - The laws of physics are the same for all reference frames
  - The speed of light is the same (c) in all reference frames
- Consequences:
  - Time dilation
    - clocks in motion run slower $\Delta t' = \gamma\, \Delta t$
  - Length contraction
    - moving objects appear contracted $L' = \frac{L}{\gamma}$
- REALLY??? Can we check this experimentally???

Application:
Cosmic Ray Muons

- Cosmic ray muons:
  - Cosmic rays are energetic particles (mainly protons) coming from somewhere in the Universe
  - When they hit the atmosphere they will produce showers of particles
  - μ are of particular interest because they are very penetrating and have a long lifetime (2.2 μs)
- Question: Can muons produced in the upper atmosphere reach the ground?
  - Input:
    - Muon's velocity = 99.99% of velocity of light c
    - Atmosphere ~20 Km thick

Application:
Cosmic Ray Muons (2)

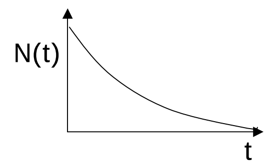
- Inputs:
  - v_μ = 99.99% of velocity of light c, atmosphere ~ 20 Km
- Non relativistic approach:
  - $\Delta l = 0.9999\ c\ \Delta t = 0.6$ Km < 20 Km: NO, they cannot reach the ground
- Relativistic approach
  - $\gamma = 1/\text{sqrt}(1-v^2/c^2) \sim 71$
  - Approach 1: our perspective
    - $\tau_\mu = 2.2\ \mu$s in muon's reference frame
    - In our reference frame: $\tau' = \tau/\gamma = 71 \times 2.2\ \mu s = 156\ \mu s$

    → Now muon can travel: Δl = 42 Km: OK!
  - Approach 2: muons' perspective
    - The $\Delta l'$ = 20 Km of atmosphere appear contracted to a relativistic μ
    - $\Delta l = \Delta l'/\gamma$ = 20Km/71 ~ 0.3 Km that can be traveled with τ=2.2 μs: OK!

Relativity: same phyiscs in all reference frames!

More on Cosmic Ray Muons

- The number of cosmic muons detected at sea level and on the top of Mount Everest are different. By how much?
- Hypotheses:
  - Muons are produced in the upper atmosphere: ~ 20 Km
  - β = 0.9999 → $\gamma = 1/\mathrm{sqrt}(1-v^2/c^2)$ ~ 71
  - Mount Everest ~ 8 Km
  - Muons decay exponentially $N(t) = N_0 \exp(-t/\tau)$



- Choose 1 RF and stay with it
  - $\tau'_\mu = 156$ μs in our R.F.
  - At sea level:
    - L=20Km → T=66 μs → $N_{sea}=N_0\exp(-66/156)=0.65\ N_0$
  - On Mount Everest:
    - L=12Km → T=40 μs → $N_{Everest}=N_0\exp(-40/156)=0.77\ N_0$

→ At sea level expect ~15% less cosmic μ than on Mount Everest: OK!

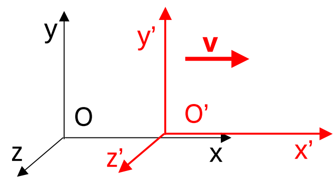
How do lengths perpendicular to v transform?

- Thought experiment
  - Train moving towards a tunnel with velocity v=0.9c
  - Height of train in train's RF: h_{train}= 3.5 m
  - Height of tunnel in tunnel's RF: h'_{tunnel} = 4.0 m
- If we have Lorentz contractions: $L'=L/\gamma$
  - $\gamma = 1/\mathrm{sqrt}(1-0.9^2)=2.29$
  - In tunnel's reference frame: the train moves with β=0.9
    → $h'_{train} = h_{train}/\gamma$ = 3.5/2.29 = 1.5m → no problem: it will fit!
  - In train's reference frame: tunnel moves with velocity β=0.9
    → $h_{tunnel} = h'_{tunnel}/\gamma$ = 4/2.29 = 1.7m < h_{train} → they will smash!

→ Different observers come to different conclusions
→ against relativity principle! → Lorentz contraction cannot happen

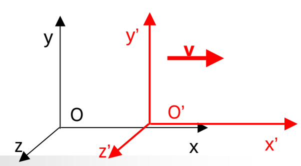
Lorentz transformation

- "Time dilation" and "Length contraction" are consequences of the so called "Lorentz transformation"
- Consider 2 inertial reference frames: O and O'
  - O' is moving w.r.t. O with velocity **v** // x axis where
    - (x,y,z,t) the coordinate in the O reference frame
    - (x',y',z',t') the coordinate in the O' reference frame



- Lorentz transformation:
  - Linear transformation that relates the coordinate in the 2 R.F.
    - Why linear? Because reference frames are inertial

Lorentz transformation (2)



- The most general form for a linear transformation:
  - z and y do not change because **v** // x
    → ignore them in the following

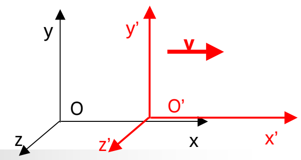
$$\begin{cases} x' = Ax + Bt & (1) \\ y' = y \\ z' = z \\ t' = Cx + Dt & (2) \end{cases}$$

- Goal: calculate coefficients A,B,C,D
- First requirement:
  - O and O' overlap at t=0: At t=t'=0, x=x'=0
  - For O, the origin of O' moves away with velocity **v** →
    Substitute in (1) $\Rightarrow\ 0 = Avt + Bt \ \Rightarrow\ \boxed{B = -vA} \ \Rightarrow \begin{cases} x' = A(x - vt) & (3) \\ t' = Cx + Dt & (4) \end{cases}$
  - For O', the origin of O moves away with velocity -**v** →

$$\text{Substitute in (3)}: \ x' = A(x - vt) \ = -Avt. \quad \text{From (4)}: \ x' = -vt' = -v(Cx + Dt) - vDt$$

$$\Rightarrow \boxed{D = A} \ \Rightarrow \begin{cases} x' = \ A(x - vt) & (3) \\ t' = \ Cx + At & (5) \end{cases}$$

Lorentz transformation (3)

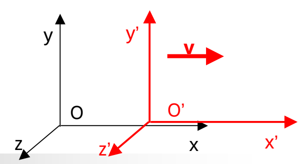


- Second requirement:
  - Send a light pulse along the x direction at t=0
  - After a time t the coordinates of the light pulse are x=ct and x'=ct'. Substitute in (3) and use (5):

$$\begin{cases} ct' = x' = A(x - vt) = A(ct - vt) \\ ct' = c(Cx + At) = c(Cct + At) \end{cases} \Rightarrow c(Cct + At) = A(ct - vt) \Rightarrow \boxed{C = -A\frac{v}{c^2}}$$

$$\Rightarrow \begin{cases} x' = A(x - vt) & (3) \\ t' = A\left(t - \frac{v}{c^2}x\right) & (6) \end{cases}$$

Lorentz transformation (4)



- Third requirement:
  - Send a light pulse along the y direction at t=0
  - After a time t the coordinates of the light pulse are (x=0; y=ct) in O; in O' the total displacement is: $x'^2+y'^2= (ct')^2$. Substitute (3) and (6):

$$x'^2 + y'^2 = (ct')^2$$

$$A^2(x - vt)^2 + y^2 = c^2A^2\left(t - \frac{v}{c^2}x\right)^2$$

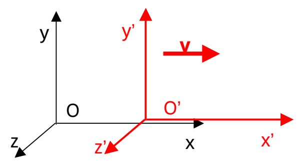
$$\text{Since } x = 0 \text{ and } y = ct \Rightarrow A^2(vt)^2 + (ct)^2 = c^2A^2t^2$$

$$\Rightarrow A = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} \equiv \gamma$$

$$\Rightarrow \begin{cases} x' = \gamma(x - vt) \\ t' = \gamma\left(t - \frac{v}{c^2}x\right) \end{cases}$$

Lorentz transformation: summary

Summarizing: when O' moves wrt O with velocity +v//x axis



- To go from O (at rest) to O' (in motion):

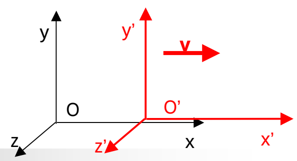
$$\begin{cases} x' = \gamma(x - vt) \\ t' = \gamma\left(t - \frac{v}{c^2}x\right) \end{cases}$$

- To go from O' (in motion) to O(at rest), just change the sign of the velocity:

$$\begin{cases} x = \gamma(x' + vt') \\ t = \gamma\left(t' + \frac{v}{c^2}x'\right) \end{cases}$$

- The other coordinates (y and z) are not affected

Transformation of velocity



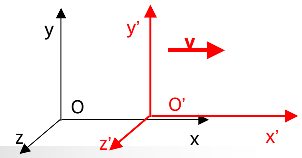
- Consequence of Lorentz transformations
- Observer in motion O' shoots a bullet with velocity u'_x // +x axis
- What is the velocity of the bullet u_x measured by O?

$$u_x = \frac{dx}{dt} = \frac{d(\gamma(x' + vt'))}{d\left(\gamma\left(t' + \frac{v}{c^2}x'\right)\right)} = \frac{dx' + v\,dt'}{dt' + \frac{v}{c^2}dx'}$$

$$= \frac{dx'/dt' + v}{1 + \frac{v}{c^2}dx'/dt'} = \frac{u'_x + v}{1 + \frac{vu'_x}{c^2}}$$

- Conclusion: $\boxed{u_x = \frac{u'_x + v}{1 + \frac{vu'_x}{c^2}}}$ and $\boxed{u'_x = \frac{u_x - v}{1 - \frac{vu_x}{c^2}}}$

Velocity not // to v



- How do we sum velocity not // to the relative motion of the 2 R.F.?
- Observer in motion O' shoots a bullet with velocity u'_y perpendicular to **v**
- What is the velocity of the bullet u_x measured by O?

$$u_y = \frac{dy}{dt} = \frac{dy'}{d\left(\gamma\left(t' + \frac{v}{c^2}x'\right)\right)} = \frac{dy'}{\gamma\left(dt' + \frac{v}{c^2}dx'\right)}$$

$$= \frac{dy'/dt'}{\gamma\left(dt' + \frac{v}{c^2}dx'\right)/dt'} = \frac{u'_y}{\gamma\left(1 + \frac{vu'_x}{c^2}\right)}$$

- Conclusion: $\boxed{u_y = \frac{u'_y}{\gamma\left(1 + \frac{vu'_x}{c^2}\right)}}$ and $\boxed{u'_y = \frac{u_y}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}}$

Summary and outlook

- Today:
  - Principle of Special Relativity and its amazing consequences
    - Length contraction and Time dilation
    - Lorentz transformations
    - Velocity transformation (v always < c)
- Next time:
  - More on Relativity:
    - How to transform electric fields and forces
    - Prove that E and B are intimately connected