8.022 (E&M) – Lecture 17

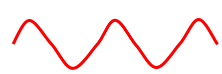
Topics:

- Discussion of Exam 2 and make-up exam
- Back to E&M:
  - RCL circuits: recap undriven RCLs, driven RCLs, inductance

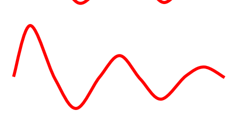
Last time

- What happens when we put inductors in circuits?
  - RL circuits: exponential solutions

  - LC circuits: oscillatory solution



  - RCL circuits: damped oscillation

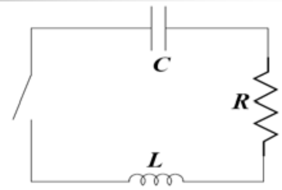


- RCL circuits are particularly interesting
  - Let's see them in some more detail…

Undriven RCL circuits: recap

- Kirchoff's second rule:

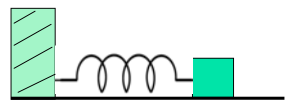
$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = 0$$



- Does it look familiar?

$$m\frac{d^2x}{dt^2} + k_f\frac{dx}{dt} + k_e\,x = 0$$

  - Mechanics: harmonic oscillator!

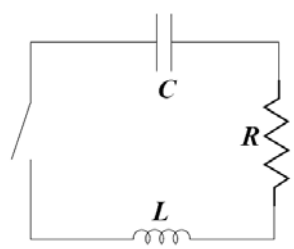


RCL	Mechanics	Interpretation
L d^2Q/dt^2	ma=m d^2x/dt^2	L ~ m: inertia term
R dQ/dt	k_f v = k_f dx/dt	R ~ k_f → friction (damping) term
1/C Q	k_e x	1/C ~k_e → elastic term due to spring

Undriven RCLs: solution

- Differential equation governing loop:

$$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{1}{LC}Q = 0$$



- Solve using complex number notation:

$$\tilde{Q}(t) = e^{\beta t} = e^{-\alpha t}e^{i\omega t}$$

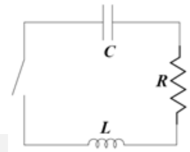
NB: $\beta = -\alpha + i\omega$ is a complex number, with α and ω real

$e^{-\alpha t}$ = damping term, $e^{i\omega t}$ = oscillatory term

Throw this into the equation and we get a quadratic equation in β:

$$\beta^2 + \beta\frac{R}{L} + \frac{1}{LC} = 0 \quad \Rightarrow \quad \beta = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

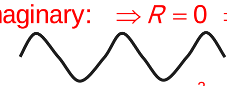
RCL circuits: solution



$$\beta = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} :$$

$$\tilde{Q}(t) = e^{\beta t} = e^{-\alpha t} e^{i\omega t}$$

- β purely real: $\frac{R^2}{4L^2} - \frac{1}{LC} > 0 \Rightarrow R > 2\sqrt{\frac{L}{C}} \Rightarrow$ 
- β purely imaginary: $\Rightarrow R = 0 \Rightarrow$ undamped LC $\Rightarrow$



- β truly complex: R>0 and $\frac{R^2}{4L^2} - \frac{1}{LC} < 0 \Rightarrow$

$\alpha = \frac{R}{2L}$ and $\omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$

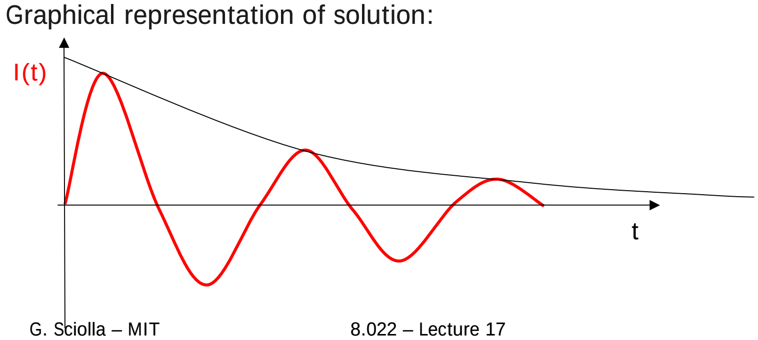
When $\frac{R^2}{4L^2} - \frac{1}{LC} = 0$ critical damping (fastest way to damp an oscillator).

RCL in weak damping limit

- Initial conditions: $Q(0)=Q_0=A\cos(\phi_0)$ and $I(0)=0=A\omega_0\sin\phi_0 \Rightarrow A = Q_0;\ \phi_0 = 0$

$$\Rightarrow \begin{cases} Q(t) \sim Q_0 e^{-\frac{R}{2L}t} \cos(\omega_0 t) \\ I(t) \sim \omega_0 Q_0 e^{-\frac{R}{2L}t} \sin(\omega_0 t) \end{cases}$$

- Graphical representation of solution:



Energy

- Energy of the circuit in the weak damping limit:

$$U_C(t) = \frac{Q^2(t)}{2C} = \frac{Q_0^2}{2C} e^{-Rt/L} \cos^2 \omega_0 t$$

$$U_L(t) = \frac{1}{2} L I(t)^2 = \frac{1}{2} \omega_0^2 L Q_0^2 \; e^{-Rt/L} \sin^2 \omega_0 t = \frac{Q_0^2}{2C} e^{-Rt/L} \sin^2 \omega_0 t$$

$$\Rightarrow U(t) = U_L(t) + U_C(t) = \frac{Q_0^2}{2C} e^{-Rt/L} (\sin^2 \omega_0 t + \cos^2 \omega_0 t) = \frac{Q_0^2}{2C} e^{-Rt/L}$$

- Since $Q^2_0/2C$=total energy stored initially in the system
→ U decreases exponentially over time: as expected!

Quality Factor

- Definition 1: the quality factor measures how many times the circuit oscillates before it loses a certain amount of energy

  In the time τ=L/R the energy decreases by ΔU(t)=1/e

  $$U(t) = \frac{Q_0^2}{2C} e^{-Rt/L}$$

  The oscillation is $\omega\tau$ radians $\Rightarrow$ $Q = \omega\tau = \frac{\omega L}{R}$

- Definition 2: the quality factor measures the ratio between energy stored (in C and L) and average power dissipated (in R)

For an oscillation with frequency $\omega \Rightarrow$ $Q = \omega \frac{\text{Energy stored}}{\text{<Power>}} = \omega \frac{L I_0^2 / 2}{R I_0^2 / 2} = \frac{\omega L}{R}$

- Q factor can be defined for any system that creates vibrations.
  - Acoustics: Q of a tuning fork is much higher than the Q of a table...

Today's goal:

Driven RCL circuits

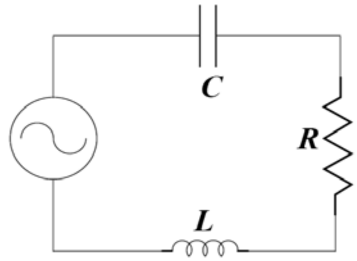
- (~) is an AC e.m.f.
- AC voltage supplied to the circuit:

$$emf(t) = V_0 \cos \omega t$$

- Convenient assumption:

$$V(t) = \text{Re}\left[\tilde{V}(t)\right] \quad \text{with} \quad \tilde{V}(t) = V_0 e^{i\omega t}$$

  - NB: V_0 is purely real!



- How to solve this? Just generalize what we used for DC!
  - Sum of voltage drops in loop is equal to emf (Kirchoff #2)

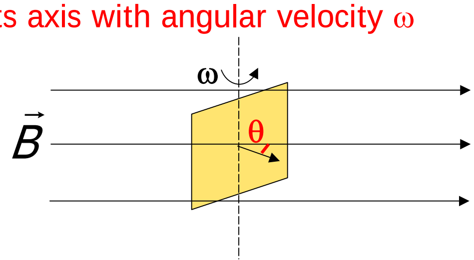
$$V_{emf}(t) = V_R(t) + V_C(t) + V_L(t)$$
$$\tilde{V}_{emf}(t) = \tilde{V}_R(t) + \tilde{V}_C(t) + \tilde{V}_L(t)$$

  - The same current must pass through every circuit element

$$I(t) = I_R(t) = I_C(t) = I_L(t)$$
$$\tilde{I}(t) = \tilde{I}_R(t) = \tilde{I}_C(t) = \tilde{I}_L(t)$$

AC current

- Consider a B constant in magnitude and a loop rotating around its axis with angular velocity ω



- If S is the area of the loop: $\int_S \vec{B} \bullet d\vec{a} = BS\cos\theta = BS\cos\omega t$
- Faraday:

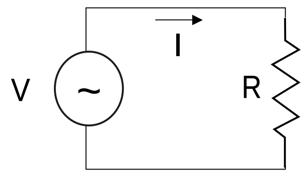
$$\left|e.m.f.\right| = \frac{1}{c}\frac{\partial}{\partial t}\left(BS\cos\omega t\right) = \frac{\omega}{c}BS\sin\omega t$$

- This is how AC power is generated. In U.S.: ν=60 Hz → ω=377

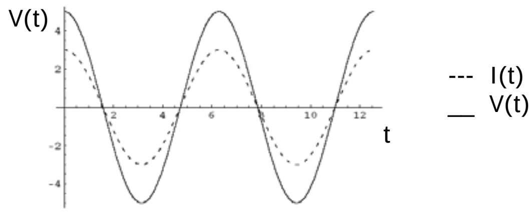
AC emf + resistor R

- Ohm's law holds for AC too:

$$V(t) = V_R(t) = I(t)R$$



- Let's plot I(t) and V(t) on the same graph:



→ In a resistor the voltage and the current are in phase

(peak voltage occurs at the same time as peak current)

Reminder: phasor notation

Any complex number $z = x + i\ y$ with $i=\sqrt{-1}$
can always be represented as the product of a real number (magnitude) and a complex exponential:

$$\Rightarrow \quad z = re^{i\theta} \quad \text{(Phasor representation)}$$

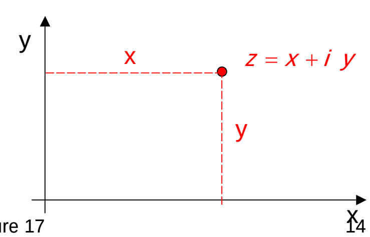
where magnitude $r=\sqrt{x^2+y^2}$ and phase $\theta=\text{arctg}\dfrac{y}{x}$

$$\Rightarrow z = r(\cos\theta + i\sin\theta)$$

and given Euler's relation:

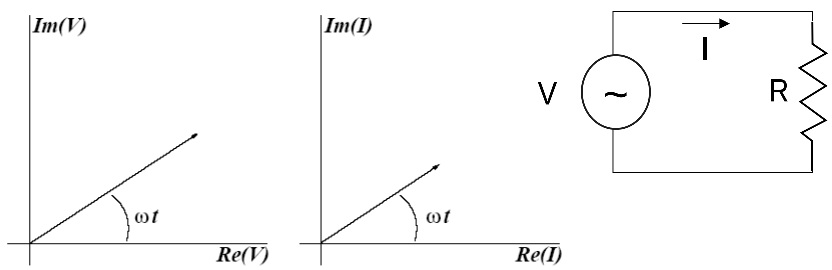
$$e^{i\theta} = \cos\theta + i\sin\theta$$

which can be easily proved using Maclaurin expansion



AC emf + R with phasors

- The same information can be represented with phasors in the complex plane: $\tilde{V}(t) = R\tilde{I}(t)$



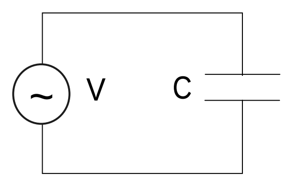
→ In a resistor the voltage and the current are in phase

In phase means that both phasors are at the same angle

AC emf + capacitor C

- Connect AC emf across a capacitor C:

$$V(t) = V_C(t) = \frac{Q(t)}{C}$$

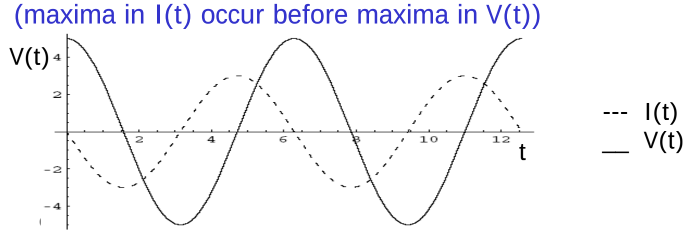


- Since $V(t)=V_0\cos\omega t$ and $I(t)= dQ/dt$:

$$I(t) = \frac{dQ(t)}{dt} = -\omega C V_0 \sin\omega t = \omega C V_0 \cos(\omega t + \frac{\pi}{2})$$

→ I(t) LEADS V(t) by 90 deg / V(t) lags I(t) by 90 deg

(maxima in I(t) occur before maxima in V(t))



Ohm's law revisited and Impedance

- Relation between I(t) and V(t) becomes more obvious when using phasor notation:

  $V_C(t) = V_0 \cos \omega t = \mathrm{Re}\left[\tilde{V}_C(t)\right]$ with $\tilde{V}(t) = V_0 e^{i\omega t}$

- For the current:

  $I(t) = \omega C V_0 \cos(\omega t + \frac{\pi}{2}) = \mathrm{Re}\left[\tilde{I}_C(t)\right]$

  with $\tilde{I}(t) = \omega C V_0 e^{i\left(\omega t + \frac{\pi}{2}\right)} = i\omega C V_0 e^{i\omega t}$ (remember: $e^{i\frac{\pi}{2}} = i$)

- Combining complex currents and voltages we can write:

  $\boxed{\tilde{V}(t) = \tilde{I}(t) Z_C}$ (complex equivalent of Ohm's law)

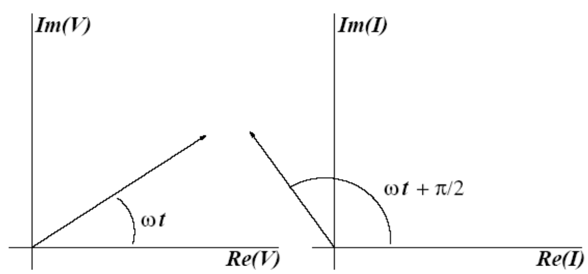
  where Z_C is the impedance of a capacitor: $\boxed{Z_C = \frac{1}{i\omega C}}$

AC emf + C: phasor representation

- Given

  $\tilde{V}(t) = V_0 e^{i\omega t}$ and $\tilde{I}(t) = Z_C V_0 e^{i\omega t} = i\omega C V_0 e^{i\omega t}$

  V(t) and I(t) can easily be represented in the complex plane:



NB: I(t) is ahead of V(t) by 90 degrees: I(t) leads V(t) by 90 degrees

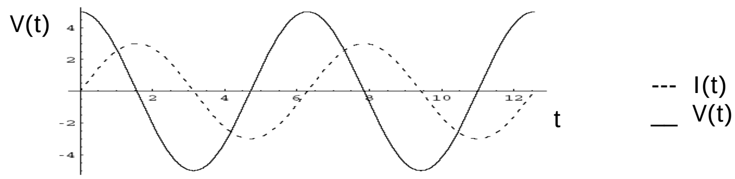
AC emf + inductor L

- Connect AC emf across an inductor L:

$$V(t) = V_L(t) = L\frac{dI}{dt}$$

- Since $V(t)=V_0\cos\omega t$:

$$\frac{dI}{dt} = \frac{V_0}{L}\cos\omega t \quad \Rightarrow \quad I(t) = \frac{V_0}{\omega L}\sin\omega t = \frac{V_0}{\omega L}\cos\left(\omega t - \frac{\pi}{2}\right)$$



→ I(t) LAGS V(t) by 90 degrees, or V(t) LEADS I(t) by 90 degrees

(maxima in I(t) occur before maxima in V(t))

Impedance of inductors

- Using phasor notation:

$$V_C(t) = V_0\cos\omega t = \mathrm{Re}\left[\tilde{V}_L(t)\right] \qquad \text{with} \qquad \tilde{V}(t) = V_0 e^{i\omega t}$$

- The current is:

$$I(t) = \frac{V_0}{\omega L}\cos(\omega t - \frac{\pi}{2}) = \mathrm{Re}\left[\tilde{I}(t)\right]$$

$$\text{with} \quad \tilde{I}(t) = \frac{V_0}{\omega L}e^{i\left(\omega t - \frac{\pi}{2}\right)} = \frac{V_0}{i\omega L}e^{i\omega t} \quad (\text{remember: } e^{-i\frac{\pi}{2}} = (i)^{-1} = -i)$$

- Combining complex currents and voltages we can write:

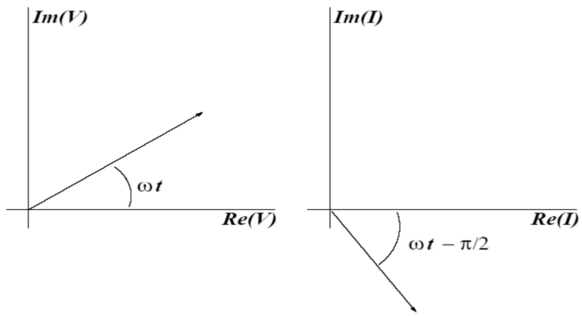
$$\boxed{\tilde{V}(t) = \tilde{I}(t)Z_L}$$ (complex equivalent of Ohm's law)

where Z_L is the impedance of an inductor: $\boxed{Z_L = i\omega L}$

AC emf + L: phasor representation

- Given $\tilde{V}(t) = V_0 e^{i\omega t}$ and $\tilde{I}(t) = Z_L V_0 e^{i\omega t} = \frac{V_0}{i\omega L} e^{i\omega t}$

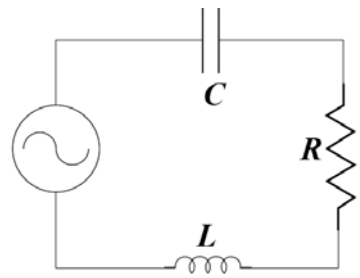
  V(t) and I(t) can easily be represented in the complex plane:



NB: I(t) is 90 degrees behind V(t): I(t) lags V(t) by 90 degrees

Driven RCLs using inductance

- Inductance simplifies the study of driven RCL circuits
- Let's work with complex numbers and use Ohm's and Kirchoff's extensions

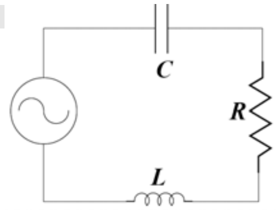


$$\tilde{V}_{emf}(t) = \tilde{V}_R(t) + \tilde{V}_C(t) + \tilde{V}_L(t)$$

Since $\begin{cases} \tilde{V}_R(t) = R\tilde{I}(t) \\ \tilde{V}_C(t) = Z_C \tilde{I}(t) = \frac{1}{i\omega C}\tilde{I}(t) \\ \tilde{V}_L(t) = Z_L \tilde{I}(t) = i\omega L \tilde{I}(t) \end{cases} \Rightarrow \tilde{V}_{emf}(t) = \tilde{I}(t)\left(R + i\left(\omega L - \frac{1}{\omega C}\right)\right) = \tilde{I}(t)\tilde{Z}_{tot}$

where total impedance of the circuit is $\boxed{\tilde{Z}_{tot} \equiv R + i\left(\omega L - \frac{1}{\omega C}\right)}$

Driven RCLs: phasor notation



- The complex current can be written as

$$\tilde{I}(t) = \frac{\tilde{V}_{emf}(t)}{Z_{tot}} = \frac{V_0 e^{i\omega t}}{R + i\left(\omega L - \frac{1}{\omega C}\right)}$$

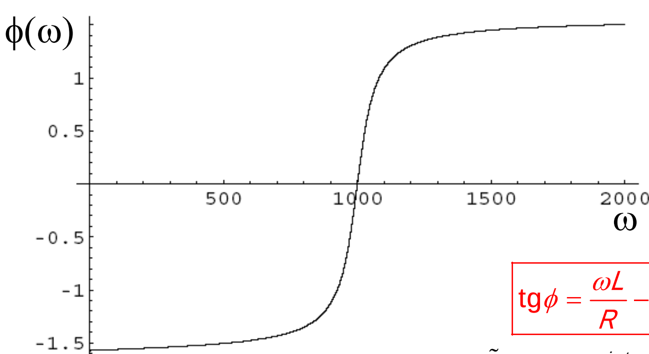
- This can be written as:

$$\tilde{I}(t) = \frac{V_0 e^{i\omega t}}{Z_{tot}} = \frac{V_0 e^{i\omega t}}{Z_{tot} Z^*_{tot}} Z^*_{tot} = \frac{V_0 e^{i\omega t}}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}\left[R - i\left(\omega L - \frac{1}{\omega C}\right)\right] = I_0 e^{i\omega t} e^{-i\phi}$$

Remembering that $e^{-i\theta} = \cos\theta - i\sin\theta \Rightarrow$

$$\begin{cases} I_0 = \dfrac{V_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \\ \text{tg}\phi = \dfrac{\omega L - \frac{1}{\omega C}}{R} = \dfrac{\omega L}{R} - \dfrac{1}{\omega RC} \end{cases}$$

Dependence of ϕ from ω



$$\text{tg}\phi = \frac{\omega L}{R} - \frac{1}{\omega RC}$$

NB: $\tilde{I}(t) = I_0 e^{i\omega t} e^{-i\phi}$

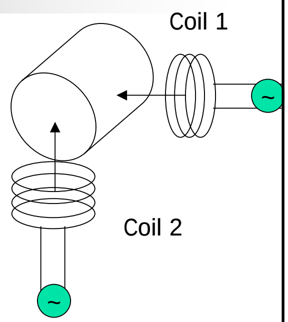
$\rightarrow$ high ω: I lags voltage by 90°

$\rightarrow$ low ω: I leads voltage by 90°

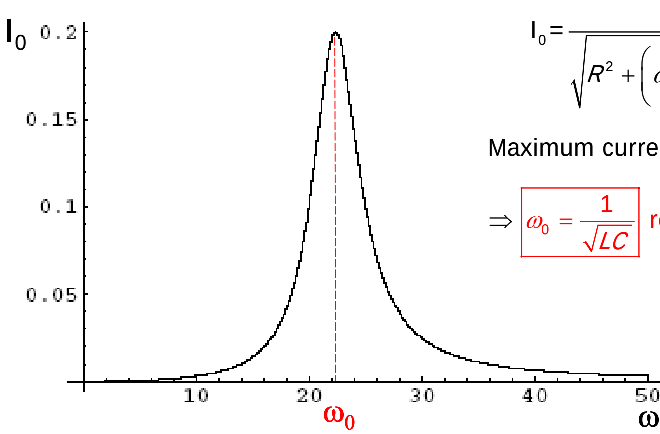
AC motor (H26)

- 2 RL circuits driven by 60 Hz AC voltage
  - Coil 1: R=2.3 Ω, L=1.5mH
  - Coil 2: R=2.5 Ω, L=31 mH
- What is the Δϕ between the 2 currents?
  - $Z_1=R_1+i\omega L_1=2.3+i\ 377\ 1.5\ 10^{-3}$
  - $Z_2=R_2+i\omega L_2=2.5+i\ 377\ 31\ 10^{-3}$

    → Δϕ=64 degrees
- The difference in phase will create a rotating B field → Eddie currents in the metal can will make it rotate!



Dependence of I_0 from ω



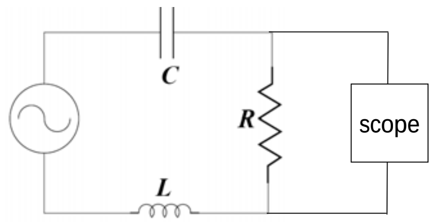
$$I_0=\frac{V_0}{\sqrt{R^2+\left(\omega L-\frac{1}{\omega C}\right)^2}}$$

Maximum current when $\omega L=\frac{1}{\omega C}$

$\Rightarrow$ $\omega_0=\frac{1}{\sqrt{LC}}$ resonance frequency

RCL resonance (Demo L8)

- RCL circuit driven with variable frequency ω
  - L=50 mH
  - C=0.3 μF

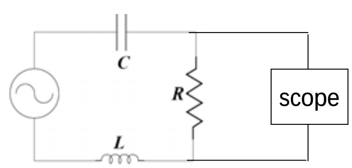


- Measure V_R on scope and tune frequency to maximize V_R
  - What is the expect resonance frequency?

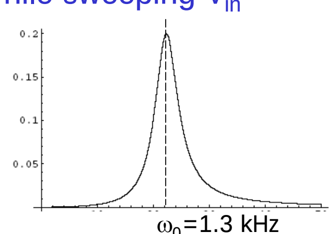
$$\omega_0 = \frac{1}{\sqrt{LC}} = 8.2 \times 10^3 \Rightarrow \nu = 1.3 \text{ kHz}$$

Demo L8: part 2

- Same RCL circuit driven with variable frequency ω
  - Frequency is driven by a voltage V_{in}
  - L=50 mH
  - C=0.3 μF

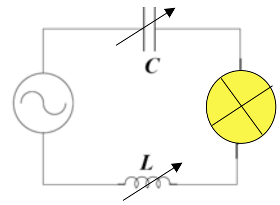


- Display V_R vs on the scope while sweeping V_{in}
  - What do you expect to see?



Resonant RCL with light bulb (L6)

- RCL circuit driven by AC voltage
  - C can be adjusted using set of switches
  - L can be adjusted moving the Fe core inside a solenoid



- For each setting of C we can find an L that turn on the light bulb
  - What is that L? $L = \frac{1}{C\omega^2}$

Summary and outlook

- Today:
  - Undriven RCL circuits
    - Energy stored and quality factor in weak damping limit
  - Driven RCL AC circuits
    - Simple solution when introducing complex impedance Z
      - $Z_R = R$
      - $Z_C = 1/(i\omega C)$
      - $Z_L = i\omega L$
- Next Tuesday:
  - More on driven RCLs: power, resonances, filters...