8.022 (E&M) – Lecture 3

Topics:

- Electric potential
- Energy associated with an electric field
- Gauss's law in differential form

... and a lot of vector calculus... (yes, again!)

Last time...

What did we learn?

- Energy of a system of charges $U = \frac{1}{2}\sum_{i=1}^{i=N}\sum_{\substack{j=1\\ j\neq i}}^{j=N}\frac{q_i q_j}{r_{ij}}$
- Electric field $\vec{E} = \frac{\vec{F}_q}{q} = \frac{Q}{|r|^2}\hat{r}$
- Gauss's law in integral form:

$$\Phi = \oint_S \vec{E} \bullet d\vec{A} = 4\pi Q_{encl}$$

  - Derived last time, but not rigorously...

Gauss's law

NB: Gauss's law only because $E \sim 1/r^2$. If E ~ anything else, the r^2 would not cancel!!!

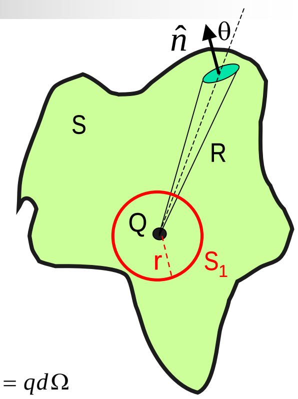
- Consider charge in a generic surface S
- Surround charge with spherical surface S_1 concentric to charge
- Consider cone of solid angle $d\Omega$ from charge to surface S through the little sphere
- Electric flux through little sphere:

$$d\Phi_{S1} = \vec{E}\bullet d\vec{A} = (\frac{q}{r^2}\hat{r})(r^2 d\Omega \hat{r}) = q d\Omega$$

- Electric flux through surface S:

$$d\Phi_S = \vec{E}\bullet d\vec{A} = (\frac{q}{R^2}\hat{r})\bullet(\frac{R^2 d\Omega}{\cos\theta}\hat{n}) = q d\Omega \frac{\hat{r}\bullet\hat{n}}{\cos\theta} = q d\Omega$$

- $d\Phi_S = d\Phi_{S1} \rightarrow \Phi_S = \Phi_{S1} = 4\pi Q$



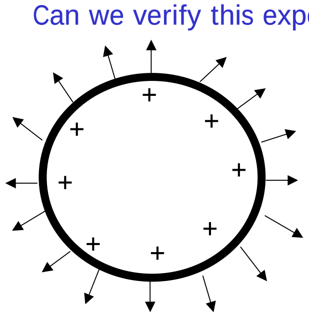
$$\Phi = \oint_S \vec{E}\bullet d\vec{A} = 4\pi Q_{encl} \text{ is valid for ANY shape S.}$$

Confirmation of Gauss's law

Electric field of spherical shell of charges:

$$\vec{E} = \begin{cases} \frac{Q}{r^2}\hat{r} & \text{outside the shell} \\ 0 & \text{inside the shell} \end{cases}$$

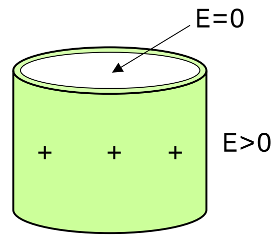
Can we verify this experimentally?



- Charge a spherical surface with Van de Graaf generator
- Is it charged? (D7 and D8)
- Is Electric Field radial? Does $E \sim 1/r^2$, eg: $\phi \sim 1/r$? Neon tube on only when oriented radially (D24)
- (D29?)

Confirmation of Gauss's law (2)

- Cylindrical shell positively charged
  - Gauus tells us that
    - $E_{inside} = 0$
    - $E_{outside} > 0$

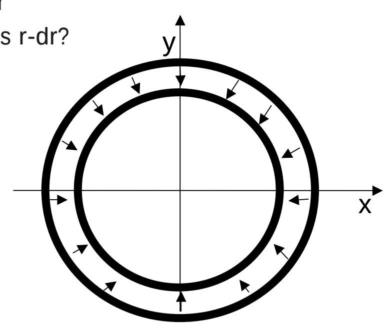


- Can we verify this experimentally?
  - Demo D26
    - Charge 2 conductive spheres by induction outside the cylinder: one sphere will be + and the other will be -: it works because $E_{outside} > 0$
    - Try to do the same inside inside cylinder → nothing happens because E=0

(explain induction on the board)

Energy stored in E: Squeezing charges...

- Consider a spherical shell of charge of radius r
- How much work dW to "squeeze" it to a radius r-dr?
- Guess the pressure necessary to squeeze it:



$$P = \frac{F}{A} = \frac{QE}{A} = E\frac{Q}{A} = E\sigma$$

$$E_{outside} = \frac{Q}{r^2}; E_{inside} = 0 \rightarrow E_{surface} = \frac{1}{2}\frac{Q}{r^2}$$

$$\rightarrow P = E\sigma = \frac{1}{2}\frac{Q}{r^2}\sigma = \frac{\sigma}{2r^2}\left(4\pi r^2\sigma\right) = 2\pi\sigma^2$$

- We can now calculate dW:

$$dW = Fdr = (PA)dr = (2\pi\sigma^2)(4\pi r^2)dr = 2\pi\sigma^2 dV$$

(where $dV = 4\pi r^2 dr$)

Remembering that $E_{\text{created in dr}} = 4\pi\sigma$ $\Rightarrow$ $$dW = \frac{E^2}{8\pi}dV$$

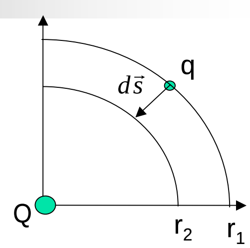
Energy stored in the electric field

- Work done on the system: $dW = \frac{E^2}{8\pi} dV$
  - We do work on the system (dW): same sign charges have been squeezed on a smaller surface, closer together and they do not like that…
- Where does the energy go?
  - We created electric field where there was none (between r and r-dr)
    → The electric field we created must be storing the energy
  - Energy is conserved → dU = dW
  - $u = \frac{E^2}{8\pi}$ is the energy density of the electric field E
- Energy is stored in the E field: $U = \int_{\text{Entire space}} \frac{E^2}{8\pi} dV$
- NB: integrate over entire space not only where charges are!
  - Example: charged sphere

Electric potential difference

- Work to move q from r_1 to r_2:

$$W_{12} = \int_1^2 \vec{F}_I \bullet d\vec{s} = -\int_1^2 \vec{F}_{Coulomb} \bullet d\vec{s} = -q\int_1^2 \vec{E} \bullet d\vec{s}$$



- W_{12} depends on the test charge q ☹
  → define a quantity that is independent of q and just describes the properties of the space:

$$\phi_{12} \equiv \frac{W_{12}}{q} = -\int_1^2 \vec{E} \bullet d\vec{s}$$

Electric potential difference between P_1 and P_2

- Physical interpretation:
  ϕ_{12} is work that I must do to move a unit charge from P_1 to P_2
- Units:
  - cgs: statvolts = erg/esu; SI: Volt = N/C; 1 statvolts = “3” 10_2 V

Electric potential

- The electric potential difference ϕ_{12} is defined as the work to move a unit charge between P_1 and P_2: we need 2 points!
- Can we define similar concept describing the properties of the space?
  - Yes, just fix one of the points (e.g.: P_1=infinity):

$$\boxed{\phi(\vec{r}) = -\int_{\infty}^{\vec{r}} \vec{E} \bullet d\vec{s}} \quad \Leftarrow \quad \text{Potential}$$

- Application 1: Calculate $\phi(r)$ created by a point charge in the origin:

$$\phi(\vec{r}) = -\int_{\infty}^{\vec{r}} \vec{E} \bullet d\vec{s} = -\int_{\infty}^{r} \frac{q}{r^2} dr = \frac{q}{r}$$

- Application 2: Calculate potential difference between points P_1 and P_2:

$$\phi_{12} = -\int_{r1}^{r2} \vec{E} \bullet d\vec{s} = \frac{q}{r_2} - \frac{q}{r_1} = \phi(P_2) - \phi(P_1)$$

→ Potential difference is really the difference of potentials!

Potentials of standard charge distributions

The potential created by a point charge is $\phi(\vec{r}) = \frac{q}{r}$

→ Given this + superposition we can calculate anything!

- Potential of N point charges: $\phi(\vec{r}) = \sum_{i=1}^{N} \frac{q_i}{r_i}$
- Potential of charges in a volume V: $\phi(\vec{r}) = \int_V \frac{\rho dV}{r}$
- Potential of charges on a surface S: $\phi(\vec{r}) = \int_S \frac{\sigma dA}{r}$
- Potential of charges on a line L: $\phi(\vec{r}) = \int_L \frac{\lambda dl}{r}$

Some thoughts on potential

- Why is potential useful? Isn't E good enough?
  - Potential is a scalar function → <u>much easier</u> to integrate than electric field or force that are vector functions
- When is the potential defined?
  - Unless you set your reference somehow, the potential has no meaning
  - <u>Usually</u> we choose ϕ(infinity)=0
    - This does not work always: e.g.: potential created by a line of charges
- Careful: do not confuse potential ϕ(x,y,z) with potential energy of a system of charges (U)
  - Potential energy of a system of charges: work done to assemble charge configuration $U = \frac{1}{2}\sum_{i=1}^{i=N}\sum_{\substack{j=1\\ j\neq i}}^{j=N}\frac{q_i q_j}{r_{ij}}$
  - Potential: work to move test charge from infinity to (x,y,z) $\phi(\vec{r}) = \sum_{i=1}^{N}\frac{q_i}{r_i}$

Energy of electric field revisited

- Energy stored in a system of charges: $U = \frac{1}{2}\sum_{i=1}^{i=N}\sum_{\substack{j=1\\ j\neq i}}^{j=N}\frac{q_i q_j}{r_{ij}}$
- This can be rewritten as follows:

$$U = \frac{1}{2}\sum_{j\neq i} q_j \sum_i \frac{q_i}{r_{ij}} = \frac{1}{2}\sum_{j\neq i} q_j \phi(r_j)$$

where $\phi(r_j)$ is the potential due to all charges excepted for the q_j at the location of q_j (r_j)

- Taking a continuum limit:

$$\boxed{U = \frac{1}{2}\int_{\substack{\text{Volume}\\ \text{with}\\ \text{charges}}} \rho\phi(r)dV = \int_{\substack{\text{Entire}\\ \text{space}}} \frac{E^2}{8\pi}dV}$$

NB: this works only when ϕ(infinity)=0

Connection between ϕ and E

Consider potential difference between a point at r and r+dr:

$$d\phi = -\int_{\vec{r}}^{\vec{r}+d\vec{r}} \vec{E} \bullet d\vec{s} \sim -\vec{E}(\vec{r}) \bullet d\vec{r}$$

The infinitesimal change in potential can be written as:

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz \equiv \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right) \bullet (dx, dy, dz) \equiv \nabla \phi \bullet d\vec{r}$$

$$\boxed{\vec{E} = -\nabla \phi}$$

Useful info because it allows us to find E given ϕ

- Good because ϕ is much easier to calculate than E

Getting familiar with gradients...

1d problem:

$$\nabla f(x) \equiv \frac{\partial f}{\partial x}\hat{x}$$

- The derivative df/dx describes the function's slope

→ The gradient describes the change of the function and the direction of the change

2d problem:

$$\nabla f(x,y) \equiv \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} \equiv \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

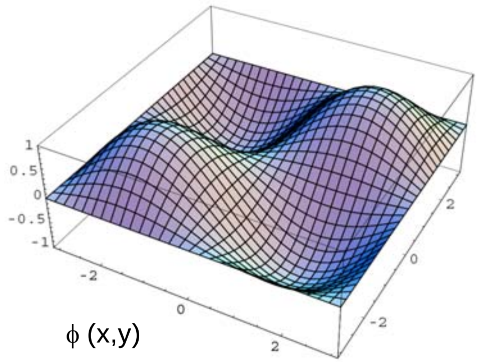
- The interpretation is the same, but in both directions

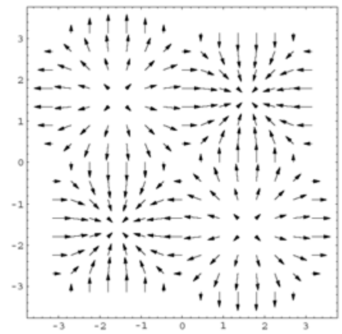
→ The gradient points in the direction where the slope is deepest

Visualization of gradients

Given the potential ϕ(x,y)=sin(x)sin(y), calculate its gradient.

$$\nabla\phi(x,y)=\cos(x)\sin(y)\hat{x}+\sin x\cos y\hat{y}$$


ϕ (x,y)

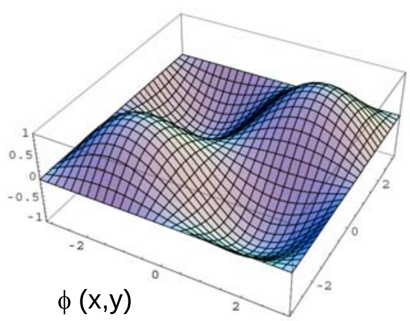

grad ϕ

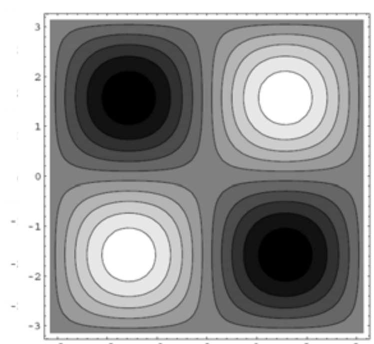
The gradient always points uphill → E=-gradϕ points downhill

Visualization of gradients:

equipotential surfaces

Same potential ϕ(x,y)=sin(x)sin(y)


ϕ (x,y)

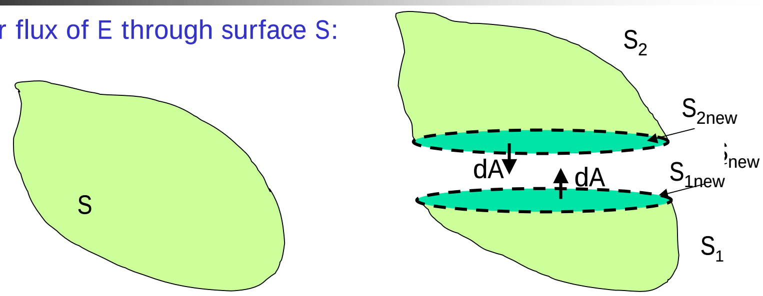


NB: since equipotential lines are perpendicular to the gradient

→ equipotential lines are always perpendicular to E

Divergence in E&M (1)

Consider flux of E through surface S:



Cut S into 2 surfaces: S_1 and S_2 with S_{new} the little surface in between

$$\Phi = \oint_S \vec{E}\bullet d\vec{A} = \oint_{S1-S1new} \vec{E}\bullet d\vec{A} + \oint_{S2-S2new} \vec{E}\bullet d\vec{A}$$

$$= \oint_{S1} \vec{E}\bullet d\vec{A} - \oint_{S1new} \vec{E}\bullet d\vec{A} + \oint_{S2} \vec{E}\bullet d\vec{A} - \oint_{S2new} \vec{E}\bullet d\vec{A}$$

$$= \oint_{S1} \vec{E}\bullet d\vec{A} - \oint_{S2} \vec{E}\bullet d\vec{A} = \Phi_1 + \Phi_2$$

Divergence Theorem

- Let's continue splitting into smaller volumes

$$\Phi = \sum_{i=1}^{i=\text{largeN}} \Phi_i = \sum_{i=1}^{i=\text{largeN}} \oint_{Si} \vec{E}\bullet d\vec{A}_i = \sum_{i=1}^{i=\text{largeN}} V_i \frac{\oint_{Si} \vec{E}\bullet d\vec{A}_i}{V_i}$$

- If we define the divergence of E as

$$\nabla\bullet\vec{E} \equiv \lim_{V\to 0} \frac{\oint_S \vec{E}\bullet d\vec{A}}{V}$$

→ $\Phi = \sum_{i=1}^{\text{largeN}} V_i(\nabla\bullet\vec{E}) \to \int_V \nabla\bullet\vec{E}dV$

→ $$\int_S \vec{E}\bullet d\vec{A} = \int_V \nabla\bullet\vec{E}dV$$

Divergence Theorem
(Gauss's Theorem)

Gauss's law in differential form

Simple application of the divergence theorem:

$$\begin{cases} \oint_S \vec{E} \bullet d\vec{A} = \int_V \nabla \bullet \vec{E} dV \\ \oint_S \vec{E} \bullet d\vec{A} = 4\pi Q = 4\pi \int_V \rho dV \end{cases} \rightarrow \int_V (\nabla \bullet \vec{E} - 4\pi\rho) dV = 0$$

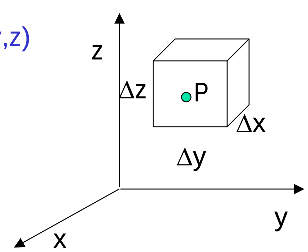
This is valid for any surface V:

$$\boxed{\nabla \bullet \vec{E} = 4\pi\rho}$$

Comments:

- First Maxwell's equations
- Given E, allows to easily extract charge distribution ρ

What's a divergence?



- Consider infinitesimal cube centered at P=(x,y,z)
- Flux of F through the cube in z direction:

$$\Delta\Phi_z = \int_{top+bottom} \vec{F} \bullet d\vec{A} \sim \Delta x \Delta y [F_z(x,y,z+\frac{\Delta z}{2}) - F_z(x,y,z-\frac{\Delta z}{2})$$

- Since $\Delta z \rightarrow 0$

$$\Delta\Phi_z = (\Delta x \Delta y \Delta z) \lim_{\Delta z \to 0} \frac{1}{\Delta z}[F_z(x,y,z+\frac{\Delta z}{2}) - F_z(x,y,z-\frac{\Delta z}{2})] = \Delta x \Delta y \Delta z \frac{\partial F_z}{\partial z}$$

- Similarly for Φ_x and Φ_y

$$\Delta\Phi_x = \Delta x \Delta y \Delta z \frac{\partial F_x}{\partial x} \quad \text{and} \quad \Delta\Phi_y = \Delta x \Delta y \Delta z \frac{\partial F_y}{\partial y}$$

Divergence in cartesian coordinates

We defined divergence as $\nabla \bullet \vec{F} \equiv \lim_{V \to 0} \frac{\oint_S \vec{F} \bullet d\vec{A}}{V}$

But what does this really mean?

$$\nabla \bullet \vec{F} \equiv \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0 \\ \Delta z \to 0}} \frac{\oint_S \vec{F} \bullet d\vec{A}}{V}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0 \\ \Delta z \to 0}} \frac{\Delta x \Delta y \Delta z (\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z})}{\Delta x \Delta y \Delta z}$$

$$= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

This is the usable expression for the divergence: easy to calculate!

Application of Gauss's law in differential form

Problem: given the electric field E(r), calculate the charge distribution that created it

$$\vec{E}(r) = \frac{4}{3}\pi K r \hat{r} \quad \text{for r<R} \quad \text{and} \quad \vec{E}(r) = \frac{4\pi K}{3r^2} R^3 \hat{r} \quad \text{for r>R}$$

Hint: what connects E and ρ? Gauss's law.

$$\oint_S \vec{E} \bullet d\vec{A} = 4\pi Q_{encl} \quad \text{(integral form)}$$

$$\nabla \bullet \vec{E} = 4\pi\rho \quad \text{(differential form)}$$

In cartesian coordinates:

$$\vec{\nabla} \bullet \vec{E} \equiv \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \ldots = \begin{cases} 4\pi K & \text{when r<R} \\ 0 & \text{when r>R} \end{cases}$$

→ Sphere of radius R with constant charge density K

Next time…

- Laplace and Poisson equations
- Curl and its use in Electrostatics
- Into to conductors (?)