**MASSACHUSETTS INSTITUTE OF TECHNOLOGY**
**Department of Physics**

# Problem Solving 1: Line Integrals and Surface Integrals

## A. Line Integrals

The line integral of a scalar function $f(x, y, z)$ along a path $C$ is defined as

$$\int_C f(x,y,z)\,ds = \lim_{\substack{N\to\infty \\ \Delta s_i \to 0}} \sum_{i=1}^{N} f(x_i, y_i, z_i)\Delta s_i$$

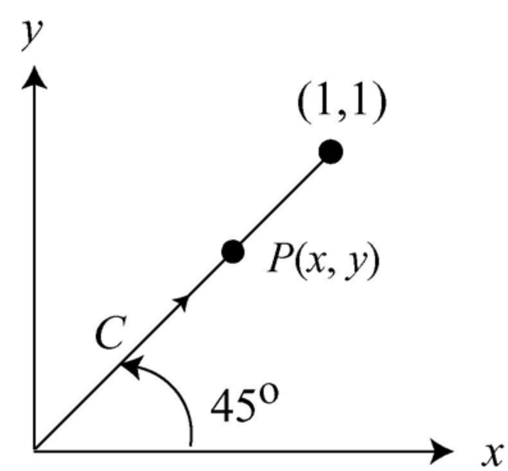
where $C$ has been subdivided into $N$ segments, each with a length $\Delta s_i$. To evaluate the line integral, it is convenient to parameterize $C$ in terms of the arc length parameter $s$. With $x = x(s)$, $y = y(s)$ and $z = z(s)$, the above line integral can be rewritten as an ordinary definite integral:

$$\int_C f(x,y,z)\,ds = \int_{s_1}^{s_2} f[x(s), y(s), z(s)]\,ds$$

**Example 1:**

As an example, let us consider the following integral in two dimensions:

$$I = \int_C (x+y)\,ds$$




where $C$ is a straight line from the origin to $(1,1)$, as shown in the figure. Let $s$ be the arc length measured from the origin. We then have

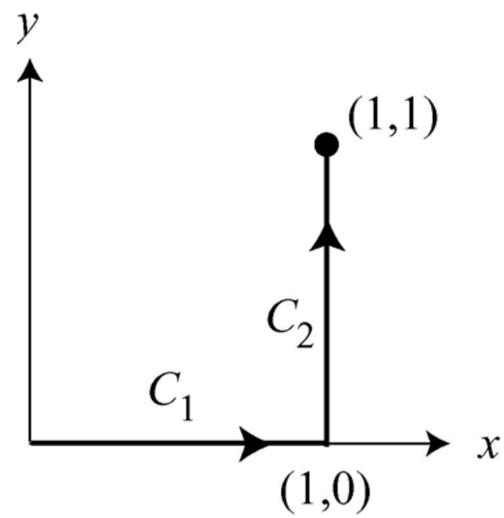
$$x = s\cos\theta = \frac{s}{\sqrt{2}}$$
$$y = s\sin\theta = \frac{s}{\sqrt{2}}$$

The endpoint (1,1) corresponds to $s = \sqrt{2}$. Thus, the line integral becomes

$$I = \int_0^{\sqrt{2}} \left(\frac{s}{\sqrt{2}} + \frac{s}{\sqrt{2}}\right) ds = \sqrt{2}\int_0^{\sqrt{2}} s\,ds = \sqrt{2}\cdot\frac{s^2}{2}\Bigg|_0^{\sqrt{2}} = \sqrt{2}$$

**PROBLEM 1:** ***(Answer on the tear-sheet at the end!)***

In this problem, we would like to integrate the same function $x + y$ as in Example 1, but along a different curve $C' = C_1 + C_2$, as shown in the figure. The integral can be divided into two parts:




$$I' = \int_{C'} (x+y)\,ds = \int_{C_1} (x+y)\,ds + \int_{C_2} (x+y)\,ds$$

(a) Evaluate $I_1 = \int_{C_1} (x+y)\,ds$.

(b) Evaluate $I_2 = \int_{C_2} (x+y)\,ds$.

(c) Now add up $I_1$ and $I_2$ to obtain $I'$. Is the value of $I'$ equal to $I = \sqrt{2}$ in Example 1 above? What can you conclude about the value of a line integral? That is, is the integral independent of the path you take to get from the beginning point to the end point?

## B. Line Integrals involving Vector Functions

For a vector function

$$\vec{\mathbf{F}} = F_x\,\hat{\mathbf{i}} + F_y\,\hat{\mathbf{j}} + F_z\,\hat{\mathbf{k}}$$

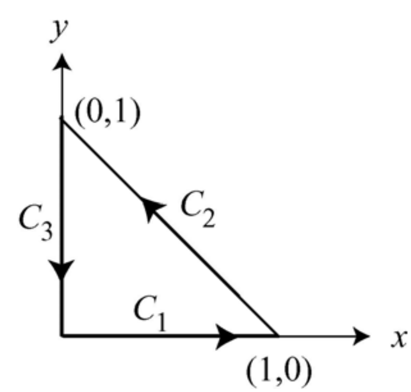
the line integral along a path $C$ is given by

$$\int_C \vec{\mathbf{F}} \cdot d\,\vec{\mathbf{s}} = \int_C \left(F_x\,\hat{\mathbf{i}} + F_y\,\hat{\mathbf{j}} + F_z\,\hat{\mathbf{k}}\right) \cdot \left(dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}} + dz\,\hat{\mathbf{k}}\right) = \int_C F_x dx + F_y dy + F_z dz$$

where

$$d\,\vec{\mathbf{s}} = dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}} + dz\,\hat{\mathbf{k}}$$

is the differential line element along $C$. If $\vec{\mathbf{F}}$ represents a force vector, then this line integral is the work done by the force to move an object along the path.




**PROBLEM 2:** ***(Answer on the tear-sheet at the end!)***

Let us evaluate the line integral of

$$\vec{\mathbf{F}}(x, y) = y\,\hat{\mathbf{i}} - x\,\hat{\mathbf{j}}$$

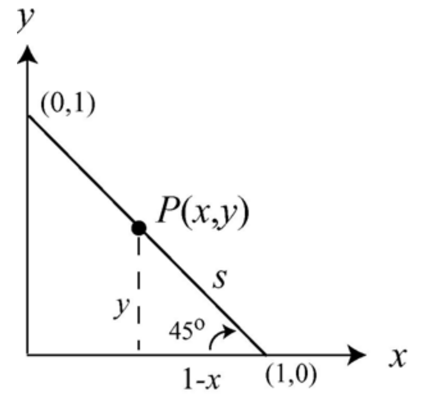
along the closed triangular path shown in the figure. Again, we divide the path into three segments $C_1$, $C_2$ and $C_3$, and evaluate the contributions separately. We will do the integral along $C_1$ for you, as follows. Along $C_1$, the value of $y$ is fixed at $y = 0$. With $d\,\vec{\mathbf{s}} = dx\,\hat{\mathbf{i}}$, we have

$$\vec{\mathbf{F}}(x,0) \cdot d\,\vec{\mathbf{s}} = \left(-x\,\hat{\mathbf{j}}\right) \cdot \left(dx\,\hat{\mathbf{i}}\right) = 0$$

So the integral along $C_1$ is zero. Now you will evaluate the integral along $C_3$. The value of $x$ is fixed at $x = 0$, $d\,\vec{\mathbf{s}} = dy\,\hat{\mathbf{j}}$, and $\vec{\mathbf{F}}(0, y) \cdot d\,\vec{\mathbf{s}} = ?$

(a) Evaluate $\int_{C_3} \vec{\mathbf{F}} \cdot d\,\vec{\mathbf{s}}$.

Finally we calculate the contribution to the line integral from $C_2$. To evaluate the integral, we again parameterize *x* and *y* in terms of the arc length *s*, which we take to be the distance between a point along $C_2$ and $(1,0)$. From the figure shown on the right, we have




$$\frac{1-x}{s}=\cos 45^\circ=\frac{1}{\sqrt{2}},\quad \frac{y}{s}=\sin 45^\circ=\frac{1}{\sqrt{2}}$$

$$x=1-\frac{s}{\sqrt{2}},\qquad y=\frac{s}{\sqrt{2}}$$

and $dx=-\frac{ds}{\sqrt{2}}$ and $dy=\frac{ds}{\sqrt{2}}$,

(b) With the information given above, evaluate $\int_{C_2}\vec{\mathbf{F}}\cdot d\,\vec{\mathbf{s}}$.

$F_x dx+F_y dy=?$

$\int_C \vec{\mathbf{F}}\cdot d\,\vec{\mathbf{s}}=\int_{C_2} F_x dx+F_y dy=?$

## C. Surface Integrals

### Double Integrals

A function $F(x, y)$ of two variables can be integrated over a surface *S*, and the result is a double integral:

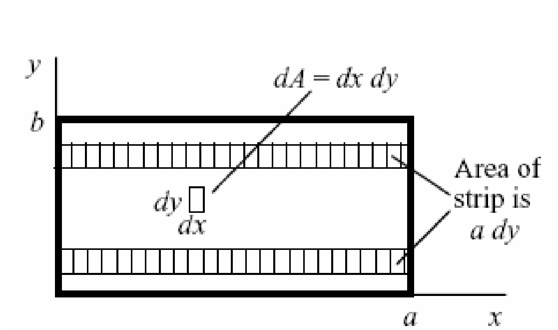
$$\iint_S F(x,y)\,dA = \iint_S F(x,y)\,dx\,dy$$

where $dA = dx\,dy$ is a (Cartesian) differential area element on *S*. In particular, when $F(x, y) = 1$, we obtain the area of the surface *S*:

$$A = \iint_S dA = \iint_S \,dx\,dy$$

For example, the area of a rectangle of length *a* and width *b* (see figure) is simply given by

$$A = \int_0^b \int_0^a dx\,dy = \int_0^b \left( \int_0^a dx \right) dy$$
$$= \int_0^b a\,dy = a\int_0^b \,dy = ab$$




Now suppose $F(x, y) = \sigma(x, y)$, where $\sigma$ is the charge density (Coulomb/m$^2$). Then the double integral represents the total charge on the surface:

$$Q = \iint_S \sigma(x,y)\,dA = \iint_S \sigma(x,y) dx\,dy$$

On the other hand, if the surface is a circle, it would be more convenient to work in polar coordinates.

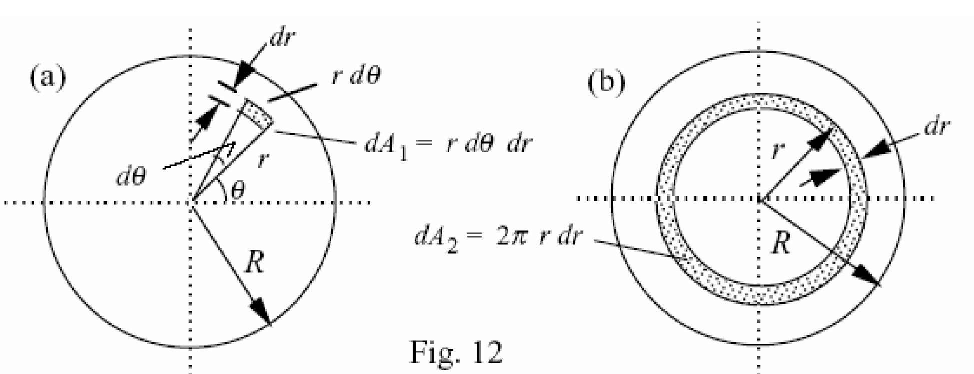



Fig. 12

The differential area element is given by (see figure above)

$$dA = r\,dr\,d\theta$$

Integrating over $r$ and $\theta$, the area of a circle of radius *R* is

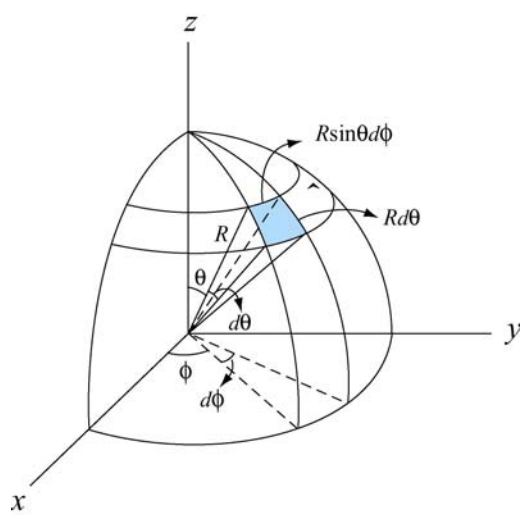
$$A = \int_0^R \int_0^{2\pi} r\,d\theta\,dr = \int_0^R \left( \int_0^{2\pi} d\theta \right) r\,dr = \int_0^R 2\pi r\,dr = 2\pi \cdot \frac{R^2}{2} = \pi R^2$$

as expected. If $\sigma(r,\theta)$ is the charge distribution on a circular plate, then the total charge on the plate would be

$$Q = \iint_S \sigma(r,\theta)\,dA = \iint_S \sigma(r,\theta) r\,d\theta\,dr$$

**Closed Surface**

The surfaces we have discussed so far (rectangle and circle) are open surfaces. A *closed* surface is a surface which completely encloses a volume. An example of a closed surface is a sphere. To calculate the surface area of a sphere of radius *R*, it is convenient to use spherical coordinates. The differential surface area element on the sphere is given by




$$dA = R^2 \sin\theta\,d\theta\,d\phi$$

Integrating over the polar angle ($0 \le \theta \le \pi$) and the azimuthal angle ($0 \le \phi \le 2\pi$), we obtain

$$\begin{aligned} A &= \oint\!\!\oint_S dA = \oint\!\!\oint_S R^2 \sin\theta\,d\theta\,d\phi \\ &= R^2 \int_0^{\pi} \sin\theta\,d\theta \int_0^{2\pi} d\phi \\ &= 4\pi R^2 \end{aligned}$$

Suppose charge is uniformly distributed on the surface of the sphere of radius *R*, then the total charge on the surface is

$$Q = \oint\!\!\oint_S \sigma\,dA = 4\pi R^2 \sigma$$

where $\sigma$ is the charge density.

**PROBLEM 3:** ***(Answer on the tear-sheet at the end!)***

(a) Find the total charge $Q$ on the rectangular surface of length $a$ ($x$ direction from $x = 0$ to $x = a$) and width $b$ ($y$ direction from $y = 0$ to $y = b$), if the charge density is $\sigma(x, y) = k\,x\,y$, where $k$ is a constant.

(b) Find the total charge on a circular plate of radius $R$ if the charge distribution is $\sigma(r, \theta) = kr(1 - \sin\theta)$ .

## D. Surface Integrals involving Vector Functions

For a vector function $\vec{\mathbf{F}}(x, y, z)$, the integral over a surface $S$ is is given by

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{A}} = \iint_S \vec{\mathbf{F}} \cdot \hat{\mathbf{n}}\, dA = \iint_S F_n\, dA$$

where $d\vec{\mathbf{A}} = dA\,\hat{\mathbf{n}}$ and $\hat{\mathbf{n}}$ is a unit vector pointing in the normal direction of the surface. The dot product $F_n = \vec{\mathbf{F}} \cdot \hat{\mathbf{n}}$ is the component of $\vec{\mathbf{F}}$ parallel to $\hat{\mathbf{n}}$. The above quantity is called "flux." For an electric field $\vec{\mathbf{E}}$, the electric flux through a surface is

$$\Phi_E = \iint_S \vec{\mathbf{E}} \cdot \hat{\mathbf{n}}\, dA = \iint_S E_n\, dA$$

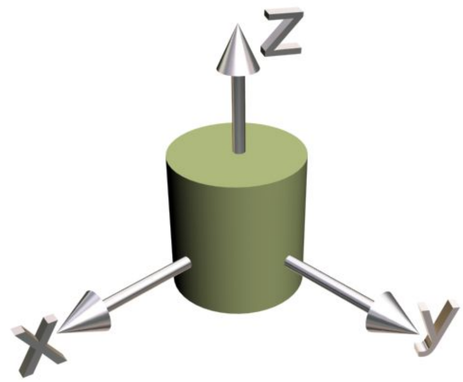
As an example, consider a uniform electric field $\vec{\mathbf{E}} = a\,\hat{\mathbf{i}} + b\,\hat{\mathbf{j}}$ which intersects a surface of area $A$. What is the electric flux through this area if the surface lies in the $yz$ plane with normal in the positive x direction? In this case, the normal vector is $\hat{\mathbf{n}} = \hat{\mathbf{i}}$, pointing in the $+x$ direction. The electric flux through this surface is

$$\Phi_E = \vec{\mathbf{E}} \cdot \vec{\mathbf{A}} = \left(a\hat{\mathbf{i}} + b\hat{\mathbf{j}}\right) \cdot A\hat{\mathbf{i}} = aA$$

**PROBLEM 4:** ***(Answer on the tear-sheet at the end!)***

(a) Consider a uniform electric field $\vec{\mathbf{E}} = a\,\hat{\mathbf{i}} + b\,\hat{\mathbf{j}}$ which intersects a surface of area $A$. What is the electric flux through this area if the surface lies (i) in the $xz$ plane with normal in the positive y direction? (ii) in the $xy$ plane with the normal in the positive z direction?

(b) A cylinder has base radius $R$ and height $h$ with its axis along the z-direction. A uniform field $\vec{\mathbf{E}} = E_o \hat{\mathbf{j}}$ penetrates the cylinder. Determine the electric flux $\iint_S \vec{\mathbf{E}} \cdot \hat{\mathbf{n}}\, dA$ for the side of the cylinder with $y > 0$, where the area normal points away from the interior of the cylinder.




Hints: If $\theta$ is the angle in the $xy$ plane measured from the $x$-axis toward the positive $y$-axis, what is the differential area of the side of the cylinder in term of $R$, $dz$, and $d\theta$?

What is the vector formula for the normal $\hat{\mathbf{n}}$ to the side of the cylinder with $y > 0$, in terms of $\theta$, $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$? What is $\vec{\mathbf{E}} \cdot \hat{\mathbf{n}}$?

$$\iint_S \vec{\mathbf{E}} \cdot \hat{\mathbf{n}}\, dA = ?$$

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**Tear off this page and turn it in at the end of class !!!!**
Note: Writing in the name of a student who is not present is a Committee on Discipline offense.

Names ______________________________________

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**PROBLEM 1:**

(a) $I_1 = \int_{C_1} (x+y)\, ds =$

(b) $I_2 = \int_{C_2} (x+y)\, ds =$

(c) $I' = I_1 + I_2 =$

Is the value of $I'$ equal to $I = \sqrt{2}$ in Example 1 above? What can you conclude about the value of a line integral? That is, is the integral independent of the path you take to get from the beginning point to the end point?

**PROBLEM 2:**

(a) $\int_{C_3} \vec{\mathbf{F}} \cdot d\,\vec{\mathbf{s}} =$

(b) $\int_{C_2} \vec{\mathbf{F}} \cdot d\,\vec{\mathbf{s}} =$

**PROBLEM 3:**

(a) Total charge $Q =$

(b) Total charge $Q =$

**PROBLEM 4:**

(a) Consider a uniform electric field $\vec{\mathbf{E}} = a\,\hat{\mathbf{i}} + b\,\hat{\mathbf{j}}$ which intersects a surface of area $A$. What is the electric flux through this area if the surface lies

(i) in the *xz* plane?

(ii) in the *xy* plane?

(b) Determine the electric flux $\iint_S \vec{\mathbf{E}} \cdot \hat{\mathbf{n}}\, dA$ for the side of the cylinder with $y > 0$.