MIT Course 8.033, Fall 2005, Symmmetry & Invariance
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1 Topics

- The notion of symmetry in physics. Key concepts:
  - Frame
  - Inertial frame
  - Transformation
  - Invariant
  - Invariance
  - Symmetry
  - Relativity
- Examples of possible symmetry:
  - Translation
  - Rotation
  - Parity
  - Galilean
  - Lorentz
  - Diffeomorphism
  - Gauge
- We will study the symmetry of:
  - Classical mechanics — both initial conditions ($\mathbf{v}$, E, *etc.*) and laws
  - Electromagnetism — in particular
    * The wave equation
    * Observed properties of light — does speed depend on wavelength? On motion of source? On motion of observer (frame)?
- Key people:
  - Galileo Galilei, 1564-1642
  - Emmy Noether, 1882-1935
  - Michaelson & Morley

2 Formula summary: transformations

- Translation:

$$\begin{cases} \mathbf{r}' = \mathbf{r} + \Delta r \\ t' = t + \Delta t \end{cases}$$

- Rotation:

$$\begin{cases} \mathbf{r}' = \mathbf{R}\mathbf{r} \\ t' = t \end{cases}$$

- Parity

$$\begin{cases} \mathbf{r}' = -\mathbf{r} \\ t' = t \end{cases}$$

- Galilean:

$$\begin{cases} \mathbf{r}' = \mathbf{r} - \mathbf{v}t \\ t' = t \end{cases}$$

- Lorentz:

$$\mathbf{x}' = \mathbf{\Lambda}\mathbf{x}, \quad \textit{i.e.}, \text{ for } \mathbf{v} \text{ in } x\text{-direction,}$$

$$\begin{cases} x' = \gamma(x - vt) \\ y' = y \\ z' = z \\ t' = \gamma(t - vx/c^2) \end{cases},$$

$$\gamma \equiv \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

3 Symmetry in physics

3.1 Glossary of key concepts

- **Frame:** A prescription for measuring the physical quantities (*e.g.*, $\mathbf{r}$, t) that appear in our equations (an operational *definition* of them).
- **Inertial frame:** A particular kind of frame in which Newton's 1st law holds.
- **Transformation:** The mathematical operation converting the quantities measured in one frame into the quantities measured in another. To avoid confusion, we usually put primes on the quantities in one of the two frames.
- **Invariant:** A quantity or equation that is left unchanged by a transformation. A quantity Q is invariant if $Q' = Q$. An equation is invariant if it stays true in the new frame (*i.e.*, when you put primes on all quantities).
- **Invariance:** Being invariant. For example, the equations of classical mechanics are said to be Galilean invariant because they are invariant under Galilean transformations.
- **Symmetry:** In physics, essentially a synonym for invariance.
- **Relativity:** Essentially a synonym for invariance and symmetry, but usually restricted to the symmetries of spacetime.

3.2 Invariance under translation

- No experiment within your lab can determine whether it's been shifted sideways. In other words, the laws of nature appear to be translationally invariant.
- Original frame: masses at $\mathbf{r}_1$ and $\mathbf{r}_2$.

$$F = \frac{GmM}{|\mathbf{r}_2 - \mathbf{r}_1|^2}$$

- Primed frame: masses at $\mathbf{r}'_1 \equiv \mathbf{r}_1 + \mathbf{a}$ and $\mathbf{r}'_2 \equiv \mathbf{r}_2 + \mathbf{a}$.

$$F' = \frac{GmM}{|\mathbf{r}'_2 - \mathbf{r}'_1|^2} = \frac{GmM}{|(\mathbf{r}_2 + \mathbf{a}) - (\mathbf{r}_1 + \mathbf{a})|^2} = \frac{GmM}{|\mathbf{r}_2 - \mathbf{r}_1|^2} = F$$

- We have time translationak invariance in time as well as in space.

3.3 Invariance under rotation

- No experiment within your spaceship can determine whether it's been rotated. In other words, the laws of nature appear to be rotationally invariant.
- If you're not cool with 3×3 matrices, please read the matrix primer handout.
- Primed frame: masses at $\mathbf{r}_1' \equiv \mathbf{R}\mathbf{r}_1$ and $\mathbf{r}_2' \equiv \mathbf{R}\mathbf{r}_2$

$$F' = \frac{GmM}{|\mathbf{R}\mathbf{r}_2 - \mathbf{R}\mathbf{r}_1|^2} = \frac{GmM}{|\mathbf{R}(\mathbf{r}_2 - \mathbf{r}_1)|^2} = \frac{GmM}{|\mathbf{r}_2 - \mathbf{r}_1|^2} = F$$

- Another example: Maxwell's equations in vacuum imply

$$\nabla^2\mathbf{E} = \frac{1}{c^2}\ddot{\mathbf{E}}.$$

  Since only *differences* in position and time enter, it's translationally invariant.
  Here it's infinitesimal differences (derivatives), above it was a finite difference $(\mathbf{r}_2 - \mathbf{r}_1)$.
- ∇^2 is invariant under rotation (remember Gauss' theorem)
- At MIT:

$$\mathbf{E} = \frac{1}{c^2}\ddot{\mathbf{E}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

- Near Australia:

$$\mathbf{E} = \frac{1}{c^2}\ddot{\mathbf{E}} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

- So both observer's agree that Maxwell was right, *i.e.*, the wave equation is translationally and rotationally invariant.

3.4 Invariance under reflection (parity)?

- Yes for all of classical physics
- Considered self-evident and obvious
- 1956: Chen Ning Yang & Tsung-Dao Lee propose that weak interactions violate parity; Chien-Shiung Wu demonstrates it with cobalt 60, Leon Lederman with accelerator. (Yang & Lee get 1957 Nobel prize.)

3.5 Symmetry is at the heart of modern physics

- Special relativity is all about so-called *Lorentz symmetry.*
- General relativity is about so-called *diffeomorphism symmetry.*
- Key topics in particle physics are C, P and T symmetry and combinations like CP and CPT symmetry.
- A cornerstone of particle physics is *gauge symmetry*
- In 2007, the Large Hadron Collider at CERN will search for *supersymmetry.*

4 The symmetry properties of classical mechanics

Under what transformations are the laws of classical mechanics invariant?
Answer: Galilean transformations.

4.1 Invariance under Galilean transformation

- Demo with colliding carts, ball.
- So Newtonian mechanics appears to be invariant - let's understand exactly what the transformation is, and why this is so.
- Inertial frame definition ($\mathbf{a} = \mathbf{0}$ if $\mathbf{F} = \mathbf{0}$)
- Are we in an inertial frame? (PS1)
- Galilean transformation definition (between 2 inertial frames)
- Definition of *event*: a 4D point (x, y, z, t). Examples?
- $\mathbf{r}' = \mathbf{r} - \mathbf{v}t$
- Lengths invariant: $\Delta\mathbf{r}' \equiv \mathbf{r}'_2 - \mathbf{r}'_1 = (\mathbf{r}_2 - \mathbf{v}t) - (\mathbf{r}_1 - \mathbf{v}t) = \Delta\mathbf{r}$
- But we must measure $\mathbf{r}'_1$ and $\mathbf{r}'_2$ at the same time!
- Which we can, since time is invariant and unambiguous: $t' = t$

4.2 Transforming velocity

- How does $\mathbf{u}$ transform under a Galilean transformation?

$$
\begin{aligned}
\mathbf{u} &\equiv \frac{d\mathbf{r}}{dt} \\
\mathbf{u}' &\equiv \frac{d\mathbf{r}'}{dt'} = \frac{d}{dt}(\mathbf{r} - \mathbf{v}t) = \frac{d\mathbf{r}}{dt} - \mathbf{v} = \mathbf{u} - \mathbf{v}
\end{aligned}
$$

  So velocities add/subtract as you'd expect: $\mathbf{u}' = \mathbf{u} - \mathbf{v}$

- But what about the flashlight on the train?

4.3 Transforming acceleration

$$
\begin{aligned}
\mathbf{a} &\equiv \frac{d\mathbf{u}}{dt} \\
\mathbf{a}' &\equiv \frac{d\mathbf{u}'}{dt'} = \frac{d}{dt}(\mathbf{u} - \mathbf{v}) = \frac{d\mathbf{u}}{dt} = \mathbf{a}
\end{aligned}
$$

So acceleration is invariant.

4.4 Transforming $\mathbf{F} = m\mathbf{a}$

- Consider forces that depend on *separation*:
  - Spring: $F = k(x_2 - x_1)$
  - Gravity: $F = \frac{GmM}{|\mathbf{r}_2 - \mathbf{r}_1|^2}$

  They are invariant, since lengths are.

- m is invariant
- Since $\mathbf{F}$, $\mathbf{a}$ and m are all invariant, so is the equation $\mathbf{F} = m\mathbf{a}$.
- So the physical law is invariant, but not the initial conditions!

4.5 Transforming energy & momentum

- Neither is invariant, since $\mathbf{v}$ isn't.
- But the conservation laws are invariant: E and $\mathbf{p}$ are conserved in any frame (PS1).
- Work-energy theorem:
$$W = \Delta KE,$$
  where work defined as
$$W = \int_{x_1}^{x_2} F dx.$$
- Proof:
$$\begin{aligned} W &= \int_{x_1}^{x_2} F dx = \int_{x_1}^{x_2} ma\, dx = m \int_{x_1}^{x_2} \frac{dv}{dt} dx \\ &= m \int_{v_1}^{v_2} \frac{dx}{dt} dv = m \int_{v_1}^{v_2} v\, dv = \frac{mv_2^2}{2} - \frac{mv_1^2}{2} = \Delta KE. \end{aligned}$$
  Only assumption here was $F = ma$, which is invariant, so the work-energy theorem is also invariant.
- W and KE alone are *not* invariant.

4.6 Transforming trajectories

- Is the 3D shape of a trajectory *not* invariant?
- No! Basket ball example: line in frame A is parabola in frame B.

4.7 Key Galilean non-invariants

$$\begin{cases} \mathbf{r}' = \mathbf{r} - \mathbf{v}t \\ \mathbf{u}' = \mathbf{u} - \mathbf{v} \\ \mathbf{p}' = \mathbf{p} - m\mathbf{v} \end{cases}$$

Quantity	Invariance		
	Translational?	Rotational?	Gallilian?
t	N	Y	Y
$\mathbf{r}$	N	N	N
Δt	Y	Y	Y
$\Delta \mathbf{r}$	Y	N	Y
$\|\Delta \mathbf{r}\|$	Y	Y	Y
d/dt	Y	Y	Y
∇	Y	N	Y
∇^2	Y	Y	Y
$\mathbf{v}$	Y	N	N
$\mathbf{p}$	Y	N	N
$\mathbf{a}$	Y	N	Y
$\mathbf{F}$	Y	N	Y
m	Y	Y	Y
E_{kin}	Y	Y	N
W	Y	Y	N
$\mathbf{F} = m\mathbf{a}$	Y	Y	Y
Newt. mechanics	Y	Y	Y
Electromagnetism	Y	Y	N

5 The symmetry properties of electromagnetism

Under what transformations are the laws of electromagnetism invariant? Let's focus on a simple special case: the laws that govern the propagation of light.

5.1 The classical wave equation

- Classical wave equation (8.03):

$$\nabla^2 \mathbf{E} - \frac{1}{c_w^2}\ddot{\mathbf{E}} = 0.$$

  For example, E could denote:

  - One of the three component of the electric field
  - One of the three component of the magnetic field
  - Air density
  - Height of water surface (2D)
  - Deflection of guitar string (1D)

- 1D special case:

$$\frac{d^2E}{dx^2} - \frac{1}{c_w^2}\frac{d^2E}{dt^2} = 0.$$

- General solution (show on PS2):

$$y = Af(x - c_w t) + Bf(x + c_w t),$$

  for arbitrary smooth function f and constants A & B.

- More complicated in 3D, but wavefronts still propagate with speed c_w.

5.2 Symmetry of the wave equation

- We learned that classical mechanics was invariant under Galilean transformations.
- The wave equation can be derived from classical mechanics.

Question: is the classical wave equation invariant under Galilean transformations?

1. Yes
2. No
3. Yes, but only if wave speed $c_w \ll c$

5.3 Transforming the wave equation

- Apply Galilean transformation to 1D wave equation:

$$\frac{d^2E}{dx^2} - \frac{1}{c^2}\frac{d^2E}{dt^2} = 0.$$

- Do this on PS2 - hints:
  - $x' = x + vt$
  - $t' = t$
  - Use chain rule for derivatives:

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial t'}{\partial x}\frac{\partial}{\partial t'} = \frac{\partial}{\partial x'} \\ \frac{\partial}{\partial t} &= \frac{\partial x'}{\partial t}\frac{\partial}{\partial x'} + \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} = v\frac{\partial}{\partial x'} + \frac{\partial}{\partial t'} \end{aligned}$$

  - Work out 2nd derivatives too

- Result:

$$\left(1 - \frac{v^2}{c^2}\right)\frac{d^2E}{dx'^2} - \frac{1}{c^2}\frac{d^2E}{dt'^2} - 2\frac{v}{c}\frac{d^2E}{dx'dt'} = 0.$$

- Wave equation *not* invariant under Galilean transformation

- Show on PS2: the new equation has solution

$$y = Af(x - [c - v]t) + Bf(x + [c + v]t),$$

  i.e., waves travel slower forward than backward.

- Just what you'd expect for waves in a substance, "aether" (velocities add).

- How can this be consistent with the wave equation being derived from classical mechanics, which is Galilean invariant?

5.4 Observed properties of speed of light

- Does speed depend on wavelength? No!
- Does speed depend on motion of source? No!
- Does speed depend on motion of observer (frame)? No!

For all three cases, let's now look at the evidence.

5.4.1 Does c depend on wavelength?

- Does light speed through glass depend on wavelength?
- But what about light speed through vacuum?
- Gamma-ray bursts provide great test
- Gamma-ray bursts last a few seconds to minutes
- Old speculations: nefarious nukes, civilization annihilation, nearby neutron stars
- Recently shown to originate at cosmological distances (few billion light years $\sim 10^{17}$ light-seconds).
- Flash seen also at x-rays and optical wavelengths, all within of order a minute $\sim 10^2$ seconds, so

$$\frac{\Delta t}{t} \lesssim \frac{10^2 \mathrm{s}}{10^{17} \mathrm{s}} = 10^{-15}.$$

- $c = d/t$, so relative speed variation with wavelength is

$$\frac{\Delta c}{c} \approx \frac{\Delta t}{t} \lesssim 10^{-15}.$$

Answer: No, at least not more than about $10^{-15} c \approx 300$ nm/s.

5.4.2 Does c depend on source motion?

- Does speed of a bullet depend on speed of rifle?
- Does sound speed of a gun shot depend on speed of rifle?
- Binary stars provide great test
- If velocities add, then

$$\begin{aligned} t_1 &= \frac{d}{c-v} \\ t_2 &= \frac{d}{c+v} \\ \Delta t &\equiv t_1 - t_2 = \frac{2dv}{c^2-v^2} \approx 2\frac{d}{c}\frac{v}{c} = 2t\frac{v}{c} \approx 200 \text{ years, say} \end{aligned}$$

  (for a pulsar in the Large Magellanic Cloud with $v = 300$ km/s, $d = 100000$ lightyears)

- But half an orbit takes only 2 days, say
- You'd see new "Doppler effect" $\propto a$ rather than v
- You'd see things moving backward in time whenever $a > \frac{c^2}{d}$ towards you

Answer: No dependence on source motion observed (and should be dramatic).

5.4.3 Does c depend on observer motion (frame)?

- No 1st order effect had been seen
- Michelson-Morley experiment hammered it - let's see how
- Consider interferometer moving with velocity $\mathbf{v}$ w.r.t. aether and compute round trip flight times parallel ($t_\parallel$) and perpendicular ($t_\perp$) to $\mathbf{v}$.
- For light traveling in direction $\pm\mathbf{v}$,

$$\begin{aligned} ct_\pm &= L_\parallel \pm vt_\pm \\ t_\pm &= \frac{L_\parallel}{cm_\text{p}v} \\ t_\parallel &= t_+ + t_- = \frac{L_\parallel}{c-v} + \frac{L_\parallel}{c+v} = \frac{2L_\parallel}{c}\gamma^2, s \end{aligned}$$

  where we have defined the quantity

$$\gamma \equiv \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}.$$

- For light traveling perpendicularly to $\mathbf{v}$,

$$\begin{aligned} (ct_\perp/2)^2 &= \sqrt{L_\perp^2 + (vt_\perp/2)^2} \\ t_\perp &= \frac{2L_\perp}{c}\gamma \end{aligned}$$

- The difference is

$$\Delta t \equiv t_\perp - t_\parallel = \frac{2L_\perp}{c}\gamma - \frac{2L_\parallel}{c}\gamma^2$$

- Rotating the interferometer by 90° changes this to

$$\Delta t' = \frac{2L_\perp}{c}\gamma^2 - \frac{2L_\parallel}{c}\gamma,$$

- *i.e.*, changes it by an amount

$$\Delta t' - \Delta t = \frac{2L_\perp}{c}\gamma^2 - \frac{2L_\parallel}{c}\gamma - \frac{2L_\perp}{c}\gamma + \frac{2L_\parallel}{c}\gamma^2 = 2\gamma(\gamma-1)\frac{L_\parallel + L_\perp}{c}.$$

- To lowest order in v/c, we have

$$\begin{aligned} \gamma &\approx 1 + \frac{1}{2}\left(\frac{v}{c}\right)^2 \\ \Delta t' - \Delta t &\approx \frac{L_\parallel + L_\perp}{c}\left(\frac{v}{c}\right)^2, \\ \frac{\Delta t' - \Delta t}{t} &\approx \left(\frac{v}{c}\right)^2. \end{aligned}$$

- $v \approx 30$ km/s, so $(v/c)^2 \sim 10^{-8}$ — tough to measure!
- But their $L_\parallel + L_\perp = 11$m was about 2×10^7 wavelengths $\lambda \sim 500$nm, and they could see fringe shifts as small as 0.01λ.
- But they saw no fringe shift at all! So c appears *not* to depend on frame.

5.5 Aether rescue attempts (see Resnick Table 1-2)

- Lorentz-Fitzgerald contraction: $L_\parallel$ contracts to $L_\parallel/\gamma$.
- Ruled out by Kennedy & Thorndike (1932) using interferometer with $L_\parallel \neq L_\perp$
- Aether drag hypothesis
- Ruled out by stellar aberration
- Also by light propagation in moving water (Fizeau 1851)
- Emission theories (v depends on source speed)
- Ruled out by binary stars (above)
- Also ruled out by Michelson-Morley with extraterrestrial light
- Also ruled out by measuring speed of γ-rays from CERN particle decays

We've seen that classical mechanics is invariant under Galilean transformations but electromagnetism isn't.

Question: What is wrong?

1. The idea that all inertial frames are equivalent
2. Our theory of mechanics (8.01)
3. Our theory electromagnetism (8.02)
4. Nothing, because of Bohr's complementarity principle

6 What are we to make of this?

- Parity symmetry applied to some things, not others.
- Is it the same with Galilean symmetry?
- **An experimental question:** Is physics the same in all inertial frames?
- A: Experiments suggest **YES**, both for mechanics and electromagnetism
- **A theoretical question:** How describe this invariance mathematically, *i.e.*,
  what is the tranformation law that leaves physics invariant?
- Galilean transformation? Works for mechanics but fails for $E\&M$
- Lorentz transformation? Works for $E\&M$ (PS3) but fails for mechanics
- No transformation works for both $E\&M$ and mechanics
- So at least one of the two must be wrong!
- Changing $E\&M$ to be have Galilean invariance is experimentally ruled out
- So let's try changing mechanics to be Lorentz invariant!
- BINGO! Not only OK with old experiments, but triumphed with new ones.

7 Derivation of the Lorentz Transformation

7.1 Battle Plan

We'll follow Einstein's approach and derive everything from two postulates:

1. The laws of physics are the same in all inertial frames.
2. The speed of light is same in all inertial frames.

Comments:

- 2 follows from 1 if we consider the speed of light one of the laws of physics.
- Einstein denoted inertial frame invariance "special relativity"
- As opposed to "general relativity", the generalization to non-inertial frames.

7.2 Inertial frames done carefully: rods & clocks

- Key concept: the *event*, a point in spacetime.
- Define coordinate system with three perpendicular rigid measuring rods
- Define time with *local* clocks
- Synchronize clocks with light pulses
- Minkowski diagram of synchronization procedure
- Minkowski diagram basics
- N.B. Don't confuse frame simultaneity with *seeing things* simultaneously: If you saw SN 1987A and a camera flash at the same time, did these two flashes go off simultaneously in your inertial frame?
- *The* time t in an inertial frame is also called *bookkeeper's time.* Don't confuse with the time when you *see* something happen.
- 1st shocker: Simultaneity is relative! Must abandon $t' = t$.

7.3 Transformation derivation, part I

- Let's define 4-vectors that have units of length:

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix}.$$

- In this course, we'll often use units were $c = 1$ (time measured in meters).
- Given $\mathbf{v}$, the new 4-vector $\mathbf{x}'$ is some function of $\mathbf{x}$ - which function?
- Translational invariance implies linearity:

$$\mathbf{x}' = \mathbf{\Lambda}(\mathbf{v})\mathbf{x} + \mathbf{x}_0,$$

for some offset $\mathbf{x}_0$ and some Lorentz matrix

$$\mathbf{\Lambda}(\mathbf{v}) = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} & \Lambda_{14} \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23} & \Lambda_{24} \\ \Lambda_{31} & \Lambda_{32} & \Lambda_{33} & \Lambda_{34} \\ \Lambda_{41} & \Lambda_{42} & \Lambda_{43} & \Lambda_{44} \end{pmatrix}.$$

- Why? Because

$$\mathbf{x}'_2 - \mathbf{x}'_1 = \mathbf{\Lambda}(\mathbf{v})(\mathbf{x}_2 - \mathbf{x}_1)$$

for linear relation — for any nonlinear (rigorously, non-affine) relation, the difference $(\mathbf{x}'_2 - \mathbf{x}'_1)$ won't depend only on the difference $(\mathbf{x}_2 - \mathbf{x}_1)$.

- Notation warning: book uses notation where 4th coordinate is t, not ct, so there things get uglier and not all Λ-coefficients are dimensionless.
- Notation warning: book uses $\mathbf{a}$, we use $\mathbf{\Lambda}$ since it's more standard these days.
- Velocity sign convention: velocity of primed frame in unprimed frame is $\mathbf{v}$, so velocity of unprimed frame in primed frame is $-\mathbf{v}$
- WLOG no translation: $\mathbf{x}_0 = \mathbf{0}$ (we can always translate later), so simply need to find the 4×4 matrix $\mathbf{\Lambda}(\mathbf{v})$
- (WLOG=without loss of generality.)
- WLOG no rotation (we can always rotate later), so $\mathbf{v} = \mathbf{0}$ case gives identity matrix:

$$\mathbf{\Lambda}(\mathbf{0}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- WLOG $\mathbf{v}$ in x-direction, since we can always rotate to make it so

- Our transformation respects rotational symmetry around the x-axis, so neither x' nor t' can depend on y or z, *i.e.*, we have $\Lambda_{12} = \Lambda_{13} = \Lambda_{42} = \Lambda_{43} = 0$.
- The x-axis gets transformed into

$$\begin{pmatrix} \Lambda_{11} & 0 & 0 & \Lambda_{14} \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23} & \Lambda_{24} \\ \Lambda_{31} & \Lambda_{32} & \Lambda_{33} & \Lambda_{34} \\ \Lambda_{41} & 0 & 0 & \Lambda_{44} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \Lambda_{11} \\ \Lambda_{21} \\ \Lambda_{31} \\ \Lambda_{41} \end{pmatrix},$$

so we have $\Lambda_{21} = \Lambda_{31} = 0$ since, by construction, the spatial part of the x-axis coincides continuously with the x'-axis.
- Consider events with $x = t = 0$. They get transformed into

$$\begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} = \begin{pmatrix} \Lambda_{11} & 0 & 0 & \Lambda_{14} \\ 0 & \Lambda_{22} & \Lambda_{23} & \Lambda_{24} \\ 0 & \Lambda_{32} & \Lambda_{33} & \Lambda_{34} \\ \Lambda_{41} & 0 & 0 & \Lambda_{44} \end{pmatrix} \begin{pmatrix} 0 \\ y \\ z \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \Lambda_{22}y + \Lambda_{23}z \\ \Lambda_{32}y + \Lambda_{33}z \\ 0 \end{pmatrix}.$$

So all events in this two-dimensional (y, z)-plane are simultaneous in *both* frames (with $t' = t = 0$), making it trivial to compare measuring rods in the two frames since their two endpoints can coincide in space and time. This implies that the 2×2 transformation matrix in this plane must be the identity matrix, *i.e.*, $\Lambda_{23} = \Lambda_{32} = 0$ and $\Lambda_{22} = \Lambda_{33} = 1$.
- An object moving uniformly with $x = vt$ in the unprimed frame remains at rest at the origin in the primed frame, so

$$\begin{pmatrix} \Lambda_{11} & 0 & 0 & \Lambda_{14} \\ 0 & 1 & 0 & \Lambda_{24} \\ 0 & 0 & 1 & \Lambda_{34} \\ \Lambda_{41} & 0 & 0 & \Lambda_{44} \end{pmatrix} \begin{pmatrix} vt \\ 0 \\ 0 \\ ct \end{pmatrix} = \begin{pmatrix} \Lambda_{11}vt + \Lambda_{14}ct \\ \Lambda_{24}ct \\ \Lambda_{34}ct \\ \Lambda_{41}vt + \Lambda_{44}ct \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ ct' \end{pmatrix}$$

so we have $\Lambda_{24} = \Lambda_{34} = 0$ and

$$\Lambda_{14} = -\beta\Lambda_{11},$$

where we've defined

$$\beta \equiv \frac{v}{c}.$$

- (When we do research using relativity, we normally use units where $c = 1$, so that we can write simply $\beta = v$.)
- Progress update:

$$\mathbf{\Lambda}(\mathbf{v}) = \begin{pmatrix} \Lambda_{11} & 0 & 0 & -\beta\Lambda_{11} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \Lambda_{41} & 0 & 0 & \Lambda_{44} \end{pmatrix}, \tag{1}$$

i.e.

$$\begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix} = \begin{pmatrix} \Lambda_{11} & 0 & 0 & -\beta\Lambda_{11} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \Lambda_{41} & 0 & 0 & \Lambda_{44} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix} = \begin{pmatrix} \Lambda_{11}(x - vt) \\ y \\ z \\ \Lambda_{41}x + \Lambda_{44}ct \end{pmatrix}. \tag{2}$$

7.4 Galileo and Einstein part ways

So far, we haven't assumed anything about the speed of light, so our results must still include both the Galilean transform and the Lorentz transform.
Let's do the Galilean first:

- Assuming that $t' = t$ gives $\Lambda_{41} = 0$ and $\Lambda_{44} = 1$.
- Assuming that measuring rods have the same length in both frames implies $\Lambda_{11} = 1$.
- This implies the Galilean transformation matrix:

$$\mathbf{G}(\mathbf{v}) = \begin{pmatrix} 1 & 0 & 0 & -\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

7.5 Transformation derivation, part II

Let's revert to equation (1) and assume that light has same speed c in both frames.

- Imagine a light flash created at $\mathbf{x} = (0,0,0,0)$ expanding with speed c in all directions, creating an expanding spherical wavefront of radii ct and ct' in the two frames. This light cone (a cone in 4D spacetime) is described by

$$x^2 + y^2 + z^2 - (ct)^2 = 0 \tag{3}$$

and

$$x'^2 + y'^2 + z'^2 - (ct')^2 = 0 \tag{4}$$

in the two frames.

- Substiting equation (2) into the last equation gives

$$\Lambda_{11}^2(x - vt)^2 + y^2 + z^2 - (\Lambda_{41}x + \Lambda_{44}ct)^2 = 0$$

- Rearranging terms:

$$(\Lambda_{11}^2 - \Lambda_{41}^2)x^2 + y^2 + z^2 - (\Lambda_{44} - \beta^2\Lambda_{11}^2)(ct)^2 - 2(\beta\Lambda_{11}^2 + \Lambda_{41}\Lambda_{44})ctx = 0.$$

- So the light cone is where this quadratic polynomial in x, y, z and t vanishes.
- This polynomial will vanish on the same cone as the polynomial of equation (3) only if the two polynomials are identical, *i.e.*, if

$$\begin{cases} \Lambda_{11}^2 - \Lambda_{41}^2 = 1, \\ \Lambda_{44}^2 - \beta^2\Lambda_{11}^2 = 1, \\ \beta\Lambda_{11}^2 + \Lambda_{41}\Lambda_{44} = 0. \end{cases}$$

- Solve:

$$\begin{cases} \Lambda_{41}^2 = \Lambda_{11}^2 - 1, \\ \Lambda_{44}^2 = \beta^2\Lambda_{11}^2 + 1, \\ 0 = \beta^2\Lambda_{11}^4 - \Lambda_{41}^2\Lambda_{44}^2 = ... = 1 - (1-\beta^2)\Lambda_{11} \end{cases}$$

  Solution:

$$\begin{cases} \Lambda_{11} = \gamma, \\ \Lambda_{44} = \gamma, \\ \Lambda_{41} = -\beta\gamma, \\ \Lambda_{14} = -\beta\gamma, \end{cases},$$

  where we have defined

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}.$$

- (We don't care about the 2nd solution with $\Lambda_{11} = -\gamma$, which corresponds to flipping the sign of t and x, "TP".)

- We're done! The Lorentz transformation is

$$\mathbf{\Lambda}(\mathbf{v}) = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix},$$

  i.e.,

$$\begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix} = \begin{pmatrix} \gamma(x-\beta ct) \\ y \\ z \\ \gamma(ct-\beta x) \end{pmatrix}.$$

- Compare to Einstein's 1905 paper

7.6 The inverse Lorentz transform

- Since $\mathbf{x}' = \mathbf{\Lambda}(v)\mathbf{x}$ and $\mathbf{x} = \mathbf{\Lambda}(-v)\mathbf{x}'$, we get the consistency requirement

$$\mathbf{x} = \mathbf{\Lambda}(-v)\mathbf{x}' = \mathbf{\Lambda}(-v)\mathbf{\Lambda}(v)\mathbf{x}$$

for any event $\mathbf{x}$, so we must have $\mathbf{\Lambda}(-v) = \mathbf{\Lambda}(v)^{-1}$, the matrix inverse of $\mathbf{\Lambda}(v)$.

- Is it?

$$\mathbf{\Lambda}(-\mathbf{v})\mathbf{\Lambda}(\mathbf{v}) = \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

i.e., yes!

7.7 Spacetime transformation summary

- Translation:

$$\begin{cases} \mathbf{r}' = \mathbf{r} + \Delta r \\ t' = t + \Delta t \end{cases}$$

- Rotation:

$$\begin{cases} \mathbf{r}' = \mathbf{R}\mathbf{r} \\ t' = t \end{cases}$$

- Parity

$$\begin{cases} \mathbf{r}' = -\mathbf{r} \\ t' = t \end{cases}$$

- Galilean "boost":

$$\begin{cases} \mathbf{r}' = \mathbf{r} - \mathbf{v}t \\ t' = t \end{cases}$$

- Combined:

$$\begin{cases} \mathbf{r}' = \pm\mathbf{R}\mathbf{r} + \Delta\mathbf{r} - \mathbf{v}t \\ t' = t + \Delta t \end{cases}$$

- Lorentz "boost":

$$\mathbf{x}' = \mathbf{\Lambda}\mathbf{x}, \quad \textit{i.e.}, \text{ for } \mathbf{v} \text{ in } x\text{-direction},$$

$$\begin{cases} x' = \gamma(x - vt) \\ y' = y \\ z' = z \\ t' = \gamma(t - vx/c^2) \end{cases}$$

- Poincaré:

$$\mathbf{x}' = \mathbf{\Lambda}\mathbf{x} + \mathbf{x}_0$$

This is the most general spacetime symmetry transformation of special relativity, with 10 parameters: $\mathbf{x}_0$ can give 3 independent translations in space and 1 translation in time), $\mathbf{\Lambda}$ can give 3 independent rotations, 3 independent boosts and also reversal of space and/or time.