Massachusetts Institute of Technology
Physics 8.03SC Fall 2016
Homework 7

Problems

Problem 7.1 (40 pts)

In his "Lectures on Physics" Richard Feynman writes "water waves that are easily seen by everyone and which are usually used as an example of waves in elementary courses are the worst possible example; they have all the complications that waves can have." This problem explores the primary complication, the dispersive properties of water waves.

Consder waves on the surface of water (also known as surface gravity waves). Assume that the deformations of the surface are influenced by gravity g (on Earth) and the surface tension T of the water (expressed in units of Newton per meter). If the density of the water is ρ and the undisturbed depth of the water is h, one can show that the phase velocity of the surface waves is given by

$$v_p^2(k) = \left(\frac{g}{k} + \frac{Tk}{\rho}\right)\tanh(kh)$$

where $k = 2\pi/\lambda$ is the wavennumber, $\tanh(kh)$ is the hyperbolic tangent, and viscosity is neglected. For water $\rho = 1$ gcm^{-3}

a. In terms of the given quantities, at approximately what wavelength $\lambda_{\text{critical}}$ does the effect of surface tension become comparable to the effect of gravity? For water at a temperature of 20°C the critical wavelength is $\lambda_{\text{critical}} = 2$ cm. What is the surface tension T of water?

b. Suppose $\lambda \gg \lambda_{\text{critical}}$ so that surface tension is negligible. What is the phase velocity and group velocity for shallow water waves (wavelength $\lambda \gg h$)? Are the waves dispersive under this condition?

c. Suppose $\lambda \gg \lambda_{\text{critical}}$ so that surface tension is negligible. What is the phase velocity and group velocity for deep water waves ($\lambda \ll h$)? Are the waves dispersive under this condition?

d. Suppoe that the water is deep $\lambda \ll h$ and the wavelengths are so short that $\lambda \ll \lambda_{\text{critical}}$ such that surface tension dominates and gravity is negligible. Such waves are called capillary waves. What is their phase velocity and group velocity? Are capillary waves dispersive?

Problem 7.2 (30 pts)

Figure 1 shows the dispersion curve for a certain medium. (Do not make any assumptions about the mathematical form of this curve.)

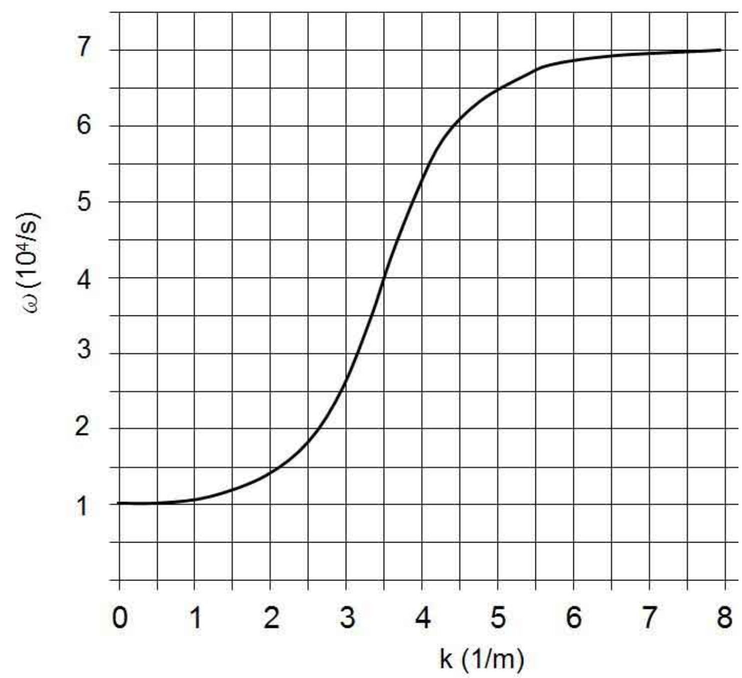

Figure 1: Dispersion relation

a. For approximately what angular frequency(ies) ω are the phase and group velocities (v_p and v_g) equal?

b. What mean angular frequency would you choose for transmitting a pulse at the highest possible speed? How big is this speed?

c. What will be the response of the medium if you excite it at $\omega = \omega_0 = 1 \times 10^4 \mathrm{s}^{-1}$? (Hint: for $\omega \to \omega_0$, what are the values of v_p and v_g?)

Problem 7.3 (30 pts)

Consider an infinite string of mass density ρ_L under tension T. A Gaussian shaped pulse is traveling on the string, causing a vertical displacement given by:

$$y(x,t) = exp\left[-\frac{1}{2}\left(\frac{(x-vt)}{\sigma}\right)^2\right]$$

a. Find the distribution of frequencies $C(\omega)$ that contribute to this pulse by calculating the "'inverse Fourier transform" (eq 10.37 in Georgi). This is a very well known integral, $C(\omega)$ is also a Gaussian with width σ_ω, feel free to use online tools to find it. Make sure that all the constants are correct.

b. Calculate the product of the widths of the two Gaussians, $(\sigma \cdot \sigma_\omega)$.

c. Sketch the shape of the pulse $y(x, 0)$ at $t = 0$ for $\sigma = 1$ and $\sigma = 5$.

d. Sketch the shape of $C(\omega)$ at $x = 0$ for $\sigma = 1$ and $\sigma = 5$ (not $\sigma_\omega = 1$ and $\sigma_\omega = 5$).

e. Compare the shapes you obtained from $y(x, 0)$ and $C(\omega)$ using different σ value. Did you see some interesting trend?