YEN-JIE LEE: Hello, everybody. Welcome back to 8.03. Today we are going to continue the discussion of waves. We will discuss a very interesting phenomenon today, which is dispersion. And before that, we will discuss a bit, just to give you some reminder, about what we have learned so far.

So we discovered this wave equation, which is showing here, in the class, and then we also show you that it described three different kinds of systems, which we included in the lecture-- the massive strings, which are the strings can actually oscillate up and down in a wide direction. And also we discussed about sound waves. This is also discussed in a previous lecture. And sound waves can be described by wave equation.

And finally, the last time we discussed electromagnetic waves. It's a special kind of wave involving two oscillating fields. One is actually the electric field, the other one is magnetic field. So that's kind of interesting, because this is actually slightly different from what we've discussed before in the previous two cases. This is actually a three dimensional wave, and also involving two different components.

And we also discussed the solution, the traveling wave solution of the electromagnetic waves. As you can see from here, the electric field is showing us the red, and the magnetic field is showing us the blue. And you can see that in case of traveling wave, they are in phase. And the magnitude reach maxima simultaneously for electric field and the magnetic field. And while in the case of standing wave, there's a phase difference. so they don't reach maxima simultaneously in the standing electromagnetic field case.

OK, so what are we going to discuss today? We would like to discuss the strategy to send information using waves. How do we actually send information using waves? So you can say, OK, maybe I can just send a harmonic oscillation. So If I do this harmonic oscillation, I can basically produce harmonic waves. They are moving up and down, and is actually always constant angular momentum and angular frequency. And maybe that's a way to send the information.

But this kind of wave is, in reality, not super helpful, because if you fill the whole space with harmonic waves, then you don't know when did you actually send the signal. Because it's always oscillating up and down, so you don't know the starting time of the signal. So in reality, these kind of simple harmonic oscillating traveling wave is not super helpful.

So what is actually helpful? That's the question. So what is actually helpful is to produce square pulse, for example. We can create square pulse, for example, in this case, I can create a square pulse here. And in the next time interval, I don't create a square pulse. In the next time interval, I don't do anything. And I create another square pulse here, et cetera, et cetera, OK.

If you use this kind of strategy what we can do is to have some kind of receiver here to actually measure the magnitude of the pulse. And then we can actually interpret this data. So this wave is going to where the positive x direction or going to the right-hand side of the board. And the receiver will be able to interpret this data by appraising this ratio on the energy or on the measure of the amplitude.

Then I can say, oh, now I'd receive a 0, and then the next signal I'm receiving is 1, and this one is 0, and 0, and 1, and 0. In this way, I can actually send information and that this information can be verified as a function of time. So in short, what would be useful is probably to use a narrow square pulse, and that would be very helpful in transmitting information.

So if we consider an ideal string case-- if I have an ideal string, as we learned before, the behavior of the string is described by the wave equation. Partial squared psi partial t squared, and this is equal to v squared partial squared psi partial t squared. And this v is actually related to the speed of the progressing wave, as we discussed before-- the progressing wave solution.

And if I have this idealized string, and it obey the wave equation, the simple version of wave equation, then I would be able to divide the dispersion relation. So I can now write down my harmonic progressing wave in the form of sine kx minus omega t. If I have a harmonic oscillating wave propagating toward the positive x direction at the speed of v. I can write it down in this functional form, where k, as a reminder, is the wavenumber, and the omega is actually the angular frequency.

And therefore, if I plug in this solution, and of course, it can have arbitrary amplitude. If I plug in this solution to this equation, then what I'm going to get is, as we did in the last few lectures, there would be a fixed relation between k, which is the wavenumber, and the omega, the angular frequency.

So the fixed relation is actually omega over k would be equal to v, which is actually the velocity in this wave equation. And from the previous discussion, we know this is actually equal to a squared root of T over rho L, where T is actually the tension, the constant tension, which we apply on this string, and the rho L is actually the mass per unit as a reminder.

So what does this mean? What does this equation mean? We call it dispersion relation a lot of time, right? But we actually didn't explain why do I do that. So we are going to learn why this is actually called this dispersion relation. Omega is a function of k. And in this case, in this very simplified idealized case, omega over k is ratio. we know this is related to the speed of propagation of the harmonic wave is equal to v. v is a constant that is independent of k. This ratio is independent of k.

What does that mean? That means if I prepare waves with different wavenumber, or in other words, waves with different wavelengths, they are going to propagate at the same speed. So the speed of the harmonic progressing wave is independent of the wavelength. That's actually very good, because in this case, if I prepared the square pulse, as we learned before, this square pulse is actually a very complicated object. Square pulse is really very complicated.

You can do a Fourier decomposition as we did before. And we need infinite number of turns of harmonic oscillating waves. We actually add them together so that I can produce a square pulse. And as I mentioned here, if the dispersion relation, omega over k, is is our constant, v. That means all the whatever wavelengths pulse, which should be added together and produce the square pulse, are going to be traveling at the that speed.

Therefore, if I have this square pulse in the beginning, after some time, t, what I'm going to get is that this is the original position of the square pulse, and after some time, t, this square pulse will move by v times t in the horizontal direction. And the shape of the pulse is not going to be changed, because no matter what kind of wavelengths which produce the square pulse, all the components in the square pulse are propagating at the same speed.

So this kind of system, which has satisfied this kind of dispersion relation is called nondispersive media. no dispersion was happening in this case, in this highly idealized case. We also know that in case of the string, we are actually making it too idealized. So if we consider a more realistic string, then I have to consider an important phenomenon, which is-- or is a important property of the string, for example-- stiffness

What do I mean by stiffness? So for example, if I take a string from a piano, a piano string, even if I don't apply any tension to the string, if I bend the string, it don't like it, all right? It's going to bounce back and restore to its original shape. So that's what I call stiffness. It's a different contribution compared to the string tension. So what we have been discussing so far that this distorting force is actually coming from the string tension, t. OK? What will happen if I introduce additional contribution from the stiffness? The stiffness is actually not completely related to the string tension, and that also wants to restore the shape of the string. OK?

Before we go to the modeling, I would like to take some votes to predict what is going to happen. How many of you were predict that if I introduce and include the stiffness of the string into my equation, will the speed of propagation increase? How many of you think it's going to happen? 1, 2, 3, 4, 5. OK. So some of you predict the speed of propagation will increase. How many of you predict that the speed of propagation of the harmonic wave will stay the same? How many of you? One? OK, only one. OK, how many of you actually predict that the speed of propagation would decrease OK so all the other students don't have opinion. OK, want to wait for the answer.

All right. So you can see that it is actually not completely obvious before we solve this question. And we are going to solve it with a simple model, which actually slightly modifies the idealized wave equation. So now, one semi-realistic model which I can introduce is to add a term additional term to the wave equation. So I can now rewrite my wave equation to include the effect that to describe a realistic string, and now this is your partial squared psi partial t squared. This will be equal to v squared partial squared psi partial t squared.

And the additional term, which I put into this, again, is minus alpha partial to the 4 psi partial x to the 4. And this is actually the contribution from the stiffness. This is stiffness. OK, so you can see that the wave equation is now modified. And what I could do in order to get the relation between omega and the k-- what I could do is that I can now start with this harmonic wave solution progressing wave solution, plug that in to this equation, this modified equation, and see what will happen.

If I plug this equation into it that modified wave equation, what I am going to get is the following. So basically the left-hand, side you're going to get minus omega squared. And then the right-hand side, you get v squared minus k squared and plus alpha k to the 4 in the right-hand side. OK, so of course, I can now cancel this minus sign. This will become plus and this will become minus. And then you can see that the relation between omega and the k is now different after I introduce this term, which is proportional to alpha. Alpha is actually describing how stiff this string is.

Of course, now I can calculate omega over k, which is actually, as we learned before, right is the speed of the propagation of a harmonic wave. So basically, if I calculate omega over k from this equation, then basically what you get is v square root of 1 plus alpha k squared. So if you look at this equation, the first reaction is, oh, now this omega and the k ratio is not a constant anymore as a function of k.

What does that mean? That means if I prepare progressing waves with different wavelengths for wavenumber k, it's going to be propagating at different speed, OK? Before we introduce this into the model, the ratio omega and k is a constant v, independent of k. Now, once you introduce this model into the equation, and you plug in the progressing wave solution to actually check the dispersion relation obtained from this equation, you'll find that the speed of progressing wave depends on how distorted this progressing wave is, OK?

So let me compare this to situation in this graph, omega versus k. So we will see this dispersion relation graph pretty often in the class today. The y-axis is actually the omega, angular frequency, and the k is the wavenumber, two pi over lambda. OK. In the original case, in the case I have this idealized string, obey the wave equation which we introduced in the previous lectures. If I plug omega as a function of k, what I'm getting is a straight line. question.

AUDIENCE: Why are you [INAUDIBLE] minus alpha [INAUDIBLE].

YEN-JIE LEE: This one, right?

AUDIENCE: [INAUDIBLE]

YEN-JIE LEE: Oh, maybe I made some mistake here. So this should be also plus here, right. So you have this-- OK, so this is omega squared, and I shouldn't have this minus sign here, right? So this should be minus, and this should be-- OK, let's go back to the original equation, OK.

So basically, you get-- so if I plug in this equation to this equation, so basically I get minus omega squared out of it. And I get minus k squared out of this. And I'm going to get plus k to the 4 out of this partial square to the 4 psi partial x to the 4. Therefore, this would be minus. OK, maybe I made a mistake. Thank you very much for spotting that.

AUDIENCE: [INAUDIBLE]

YEN-JIE LEE: Oh, yeah, I'm sorry. Not my best day today.

AUDIENCE: [INAUDIBLE]

YEN-JIE LEE: Yeah. Well, then I do it. OK. I must be drunk today.

[LAUGHTER]

Thank you very much. Anymore mistake? OK. Fortunately not, right? OK. Very good. So let me do this again. So now I can modify my wave equation, right? Originally, the wave equation is partial squared psi partial t squared equal to v squared partial squared psi partial x squared. And now I add additional term, which is actually proportional to the partial to the 4 psi partial x to the 4. OK, if I add this term into again. And now I plug in the wave equation, the progressing wave solution, into this equation, and I would get this formula, OK?

So now everything should be correct, and I have clear evidence that everybody's following. So that is very good. And now, I can now cancel all the minus sign, right and then it's become plus. And now I can actually calculate what would be the speed of propagation for this specific harmonic progressing wave and omega over k will be equal to v square root of 1 plus alpha k squared. OK?

Thank you very much for the contribution. And then now we see that here this ratio depends on k. So if I plug this on top of the previous curve, which is actually obtained from here, then what I'm going to get is something like this. In the beginning it's pretty close to the nondispersive case. And it goes up, because of this alpha contribution. Alpha is actually a positive number in my model. And the k is actually the wavenumber.

So what is going to happen is that basically after you include stiffness, the slope of this curve is changing as a function of k. OK? What do I learn from this exercise is that if I increase k, if I have a very large k-- that means I have a very small lambda, because k is actually 2 pi over lambda. OK? So that means I'm looking at something really distorted like this.

Both string tension and the stiffness wants to restore the string back to normal. Therefore, what is happening is that you are going to get additional restoring force. Therefore, as we actually calculate to here if alpha is actually positive, then the velocity actually increased with respect to what we actually get before we actually had this into a model.

So I think that makes sense, because the stiffness also wants to restore the distortion. Therefore, you have larger and larger restoring force. Therefore, the speed of propagation of this harmonic wave will increase. so that's pretty nice. But what does that mean to our project? OK, our project is to send information from one place to the other place, right? So what we just discussed is that we can actually send a square pulse and let it propagate. A square pulse can be decomposed into many, many pieces-- many, many harmonic waves. OK?

Before the square pulse works, because all the waves with different wavelengths should be moving at this constant speed, independent of the wavelengths. Now we are in trouble. As you can see here, now the speed, which is omega over k depends on the wavenumber or wavelength. Therefore, different components, which actually are needed to create a square pulse, are going to be propagating at different speed.

You can say, oh, come on, this is actually mathematics, so I don't believe you. A square pulse is a square pulse, and that's mathematics, that's math department. But we can actually really see that in the experiment. OK? So that's-- take a look at this demonstration. Maybe you didn't notice that before, but we have seen this effect from the previous lectures. OK, so I can now create a square-- not really a square pulse, but actually some kind of pulse. OK I can create some kind of pulse like this. OK?

And as we learned before, when this pulse pass through an open end, it's going to be bounced back. so therefore, I can have-- I can actually show you this demo in a limited set-up. But this pulse is going to be going back and forth, because I have open end, as we've discussed before.

What is going to happen is that since we have a realistic system, what is going to happen is that this pulse will become wider and wider, right? That's the prediction coming from this equation. Different component with different wavelengths is going to be propagating at different speed. Therefore this pulse is going to become wider, and we can see that. OK, so let me quickly produce a pulse and see what will happen. OK.

Originally, it's actually really sharp. And you can see that really the width of the pulse become wider and wider. And at some point, it disappear. If I have a very long set-up, what you are going to see is that it's going to be propagating toward the same direction. And the width of the pulse is actually going to be increasing as a function of time.

Let's take a look at this again. Now, this time we have a negative pulse. You sort of see-- very similar, see. And also you can see that there are some strange vibration actually left behind the main pulse. So that means harmonic waves with different wavelengths really propagating at different speed.

And for that, to demonstrate this effect, I also prepared some demonstration, which actually are based on our calculation, OK. So you can say that, OK, now I'm convinced I can see dispersion in the experiment. How do I know this calculation actually match with the experimental data, right? How about we really run a simulation and see what would happen.

So what this example actually do is, in the beginning, you would do integration like crazy in order to get all the components calculated. Then it's going to propagate all those pulses-- all those pulse with different components through the medium, OK? And then there will be two different colors, one is actually blue, which is the original shape. The other one is actually the one which is stiffness turned up.

So now, in the beginning I can set the alpha value to be 0.02 and see what will happen. And I will put produce triangular pulse. You can see that now. The program is really working very hard to capture all the components from 1 to 199 and equal to 1 until 99. And then now, these individual components are propagating through the medium. And you can see that originally the shape is like-- the blue shape-- triangular shape. And you can see that is a function of time. The pulse become wider and wider, OK?

Now, of course, I can increase the alpha to 0.02 and see what happen-- from 0.02 to 0.2 and see what will happen. You should expect a much larger dispersion. And you can see that now in the beginning, it's doing the integration. And you can see that this time because the alpha is actually larger. Therefore, you see that this effect, this broadening, is actually happening earlier, and it become broader and broader, and that there are a lot of strange structures, as you can see also from the demo, produce because different components are actually propagating at different speeds.

So of course, we are MIT, so in this course we have MIT-- MIT waves. So let's take a look at the MIT wave and see what will happen. Now you see that there are very sharp edge, which actually require really a lot of effort to reproduce that. And you can see that MIT is kind of distorted as a function of time. We can kind of still identify the peak, but it's actually now displaced. And in the end of the simulation, you can not even recognize that's actually originally MIT signal, which was sent from your source.

So what I want to say is that this effect, this dispersion effect, is really an enemy, which is actually very dangerous. And that actually will prevent us from sending high quality signals. OK, any questions about all those demos? Yes.

AUDIENCE: Why do we model the [INAUDIBLE]?

YEN-JIE LEE: So this is because the stiffness is actually symmetric, right. So if you bend the string, then there are contribution from the positive and negative part, OK? If you have partial to the 3, partial to the x to the 3 component, then it's going to be a symmetric and so actually against our physics intuition. And also, in this modeling, you also match with our experimental data pretty well. OK. Very good question.

And on the other hand, we now consider then the stiffness. you can also go back to the infinite number coupled oscillator case. If you instead take an example which is actually not super small displacement approximation, you take the next to leading order term. Then you will see that the partial to the 3 partial x to the 3 term as you cancel because it's symmetric, or so I argued. And then you will be able to also obtain this tern when you have slightly larger displacement with respect to the equilibrium position. So I hope that answers your question. Any other question? Yes?

AUDIENCE: If you were looking at [INAUDIBLE], for example, what would be [INAUDIBLE]?

YEN-JIE LEE: When you pass through the medium.

AUDIENCE: So [INAUDIBLE]

YEN-JIE LEE: A molecule can actually change the speed of different wavelengths, actually, differently, right? Very good question. OK, so very good.

We got two questions, and we can see that if I now turn on the alpha and make the alpha value large, then you can see that the information is distorted. And this involve infinite number of terms. And in this case, in this new demo which I show here, I have alpha value equal to 0.2. Therefore, the effect of dispersion is actually much larger than what you showed before.

And then you can see that this MIT wave quickly become something like a Gaussian-like wave, right?

OK, so very good. So you can say, OK, you are making an example-- it's a very interesting example, but it involve too many terms. You have infinite number of progressing waves in this example. It's very difficult to understand. How about we go back to a much simpler example, OK? What we can do is that instead of going through infinite number of harmonic waves, now we just consider two waves, and overlap these two waves together and see what will happen. And let's see what we can learn from it, because the required number of harmonic wave to describe such a pulse is too complicated.

So you can say that, OK, now let's just consider two waves and see what we can learn from this. And this is actually what I am going to do now. So from Bolek's lecture I hope that he covered the beat phenomenon. So basically, what is it? A beat phenomenon? Beat phenomenon happens when you overlap two waves, two harmonic waves. They have pretty close wavelengths. OK, but they're not the same.

And now, if you add two waves together, that's actually what you are going to get. You are going to get something which is oscillating really, really fast, which is basically called the carrier. And also you can see that the magnitude of the oscillation is actually changing as a function of position, and that we call envelope. So that's essentially the beat phenomenon, which you learned from previous lectures.

So in this example, I'm going to add two waves together. So the first wave is described by-- OK, is denoted by side one. It's a function of x and t, and it has a function of form A is the amplitude. And the sine k1 x minus omega1 t. This is actually a progressing wave propagating toward the right-hand side of the board, the positive direction of the x-axis in my coordinate system. And it has a wavenumber of k1 and angular frequency omega1

And I can also write down my second wave, which I would like to overlap with the first wave. So this is actually having exactly the same amplitude, which is A. And it is described by a sine function, and you have a wavenumber k2 x minus omega2 t, angular frequency omega2. With these two equations, we can calculate the speed of propagation for the individual waves, right? So the first one, I can calculate the speed of propagation v1 would be equal to omega1 over k1. Very similarly, you can also calculate the speed of propagation for the second wave, which is omega2 over k2.

So now what I'm going to do is to calculate a sum of these two waves. So I have the total, which is psi is equal to psi1 plus psi2. So what I'm going to do is to overlap these two waves and see what will happen. And for that, I need this formula, which is a sine A plus sine B. And this would be equal to 2 times sine A plus B over 2 and sine-- it would become cosine here-- cosine A minus B over 2.

So if I use that formula, basically what I'm going to get is-- we have two times from the formula. So if you have 2A sine k1 plus k2 over 2x minus omega1 plus omega2 over 2. So basically, the first term is the sine function. The sine function and the content is actually A plus B. Therefore, you add these two together, divide it by two, then basically this is as actually what you obtain.

The second term is a cosine term. You get a cosine here. But now you calculate A minus B, which is this term minus that term divided by 2. Then basically what you get is k1 minus k2 divided by 2 times x minus omega1 minus omega2 over 2 t. OK, so now this actually-- what would happen if you add these two waves together? Until now, everything is exact. And I would like to add additional conditions or additional assumptions when I discuss this solution. OK?

So how about in order to produce the beat phenomenon, I need to make the wavelengths very, very similar between the two waves. So therefore, what I am going to do is that I'm going to assume k1 is very close to k2 is roughly k. And because of this, since I have a continuous function, if k1 is really close to k2, that means omega1 is going to be also very close to omega2, right? So what I'm going to get is omega1 is going to be also very similar to omega2, and I will call it omega.

So if I do this, when I have very similar k1 and k2, what is going to happen? What is going to happen is that k1 minus k2 will be very small. So this very small k means larger wavelengths. Therefore, this cosine term will become the envelope, because it's actually a slowly variating amplitude as a function of position, because the k is very small. K is small means lambda large. Therefore, the amplitude is going to be having this modulation, which is actually like the envelope, that the oscillation of this envelope is actually controlled by the k, okay?

Let's look at the left-hand side term. k1 plus k2 over 2 is kind of like calculating the average of the wavenumber of the first and second wave. So if you calculate our average, you can be still pretty large. Therefore, you have small lambda compared to the difference. Therefore, you see that that actually contribute to those little structures in this graph, and it's called carrier. Yes?

AUDIENCE: [INAUDIBLE]? If k1 were a lot bigger than k2, then [INAUDIBLE].

YEN-JIE LEE: So they can be different. Yeah, so you are absolutely right. So you can produce something like a carrier even when k1 is not equal to k2, right? Its just a average. You're right. But then on the other hand, the difference, k1 and k2 will be also large. Therefore, it's not as easy as what we have been doing here to identify who is the carrier and who is the envelope. But you do get some kind of graph, which is oscillating really fast, but the envelope is going to be also oscillating very fast. That is harder to see all the structure. But you're absolutely right, yes. Very good question.

So now I have this set-up. I assume that they are very close to each other. So now I can define phase velocity. Finally, we define what is actually the phase velocity. In The phase velocity-- I call it vp-- you can see that before I already have been using phase velocity vp for the previous discussions. In the case of nondispersive medium, the phase velocity is just a vp, which is the velocity in the equation. And in this case, vp will be equal to omega over k, as we discussed before. And that's actually the definition of this phase velocity.

And I can now also define the group velocity. The group velocity is actually the velocity of the envelope. I can calculate the velocity of the envelope. in the case of phase velocity, I'm calculating the velocity of the carrier. I'm taking a ratio of the average, and actually the average is so close to k and omega, therefore the phase velocity vp would be just the speed of the propagation of the carrier, which is actually omega over k. I call it vp. And in case of group velocity, I call it vg. vg is describing the speed of propagation of the envelope. Therefore, what I am getting is omega1 minus omega2 divided by k1 minus k2. Both of them have effect of 1 over 2, which we say is canceled. And when they are really so close to each other, this is actually roughly like d omega dk. Any questions so far?

So we have derived two different kinds of speed. One is actually related to the phase velocity, which is-- one is actually called the phase velocity. It's related to the speed of the carrier. The other one is group velocity, which is actually related to the speed of the envelope. So let me describe you a few interesting examples. And let's see what we can actually learn from this.

In the first example, I'm working on a non dispersive medium, OK? If I have a nondispersive medium, then basically what I'm going to get is that omega will be proportional to k. If I plot omega versus k, it's a straight line. Now, if I have omega-- I choose the omega of the two, omega1, omega2, of the two waves-- to be roughly equal to omega 0, I can now evaluate the vp. The vp will be the slope of this point on the slope of a line connecting the 0 to that point, which is actually the omega over k, right? So that's actually the definition of the phase velocity. I would get this slope. The slope of this line is actually related to the phase velocity.

I can also calculate the slope of a line cuts through this point. But as it cuts through this curve, and in this case, I'm also going to get a line overlapping with phase velocity, because in this case, omega over k is a constant, which is v. Therefore, no matter what you calculate, if you calculate vp as a ratio of omega and a k, where you calculate vg, which is actually the slope of the line cutting through that point, is you always get actually v.

Therefore, what we learned from here is that for a nondispersive medium, vp will be equal to vg. That means both of these two curves, both of the curve of envelope, describing the envelope and then describing the carrier, is going to be propagating at the same speed. OK, any questions? So the whole thing is going to be moving at constant speed. For that, I can now show you some example, which I prepared, some simulation which I prepared.

So what it does is that it really-- oh, wait a second. This is 0. OK, so this is the case when I have a nondispersive medium. if I have a nondispersive medium, what is going to happen is that both the carrier, which is the speed of all those little structures, and the envelope is going to be propagating at the same speed. So you can see the high is like a fixed pattern. It's propagating toward the right-hand side. And the relative motion between the defined structure and the envelope is actually 0. So basically you have exactly the same pattern as a function of time.

So now I'm going to move away from the nondispersive medium. How about we discuss what would happen if we have considered the stiffness of the string and see what we get from there. So if I plugged omega as a function of k, and consider alpha to be non-zero. It's a positive value. So if I have alpha to be a positive value, non-zero, in this case, I'm going to get a curve like this. The slope is actually changing and it's curving up because if you have k large, then you would see that the ratio of omega and k actually increase.

So this is actually the kind of curve which we would get if I set the omega of the first and second wave of interest in this study to be omega 0. Then basically, what you are going to get is that-- OK, now I have this point here on the curve. If I calculate the phase velocity-- the phase velocity, how do I calculate that? I can now connect 0 and the point by a line. And I can now calculate the slope of this line, and I can get the phase velocity, vp.

On the other hand, I can also calculate the slope of a line cutting through, tangential to the point of interest. And that is going to give me the group velocity. As you can see from here, which slope is actually larger? Anybody know? Can point it out? Group velocity's larger, right? So in this case, if I turn on alpha greater than 0, what is going to happen is that, since the group velocity is larger than the phase velocity, that means, if I go back to that picture, the envelope is going to be moving faster than the fine structure inside the envelope.

How about we take a five-minute break from here? And then we continue the discussion after the break. It's a good time to take a break.

Welcome back, everybody. So we will continue the discussion of the beat phenomenon. So what we have shown you is that, based on those curves, actually can actually determine what will be the relative velocity-- what would be the velocity of the carrier, which is actually denoted by vp, and the what would be the velocity of the envelope, which is actually denoted by our group velocity. And in this case, what I'm actually plotting here is that, in this case, because alpha is actually greater than 0, therefore, this curve is actually curving up. Therefore, you have larger group velocity compared to the phase velocity.

So what you would expect is that the envelope is going to be actually progressing at a speed higher than the speed of the carrier. On the other hand, if magically I can construct some kind of medium which can be described in this situation, alpha smaller than 0, what is going to happen? So if I plot a situation with alpha smaller than 0, so now I plot omega was a function of k. What is going to happen is that this-- so basically, you have something which is actually curving downward.

So if I now, again, work on some point of interest here, you can see that the slope of the phase velocity is now actually larger than the slope, which is actually from the line cutting through the-- tangential to the curve, which is actually getting you the group velocity. So in the case of alpha's more than 0, which is some strange medium I can which I can create from whatever, plasma, or some really strange new kind of material of interest. If that happens, then that means your group velocity will be smoother than the phase velocity.

And if you look at this point here, you can see that this curve actually reach a maxima here. And if you actually are operating at this point, what is going to happen? What is going to happen is that if you calculate the group velocity, what will be the value? It will be 0. What does that mean? That means the envelope will not be moving a lot, but the carriers are still moving. So at this point, you are going to get group velocity equal to 0.

And finally, the if you actually going to a very large k value in this scenario, alpha smaller than 0, you will see that even you can have phase velocity, vp, positive, because it's actually a positive slope. And that the group velocity actually is negative. What does that mean? That means you are going to see a situation that the carriers are progressing in the positive direction, and the envelope is going to be progressing in the negative direction, probably progressing to the left-hand side of the board.

So what does that mean? That means this wave is doing what Michael Jackson's doing. It's actually doing the moonwalk.

[LAUGHTER]

So this is actually the kind of thing which could have happened, that it looks like and that you are doing-- going forward, because all the carriers are moving in a positive direction. But the body is actually going toward negative direction. maybe I can also learn moonwalk at some point.

[LAUGHTER]

OK. So let's go back to the demonstration which I got started, and somehow I got messed up. So let's take a look at the demo again so let's look at all the different situation at once. So in this case, as we discussed before, this is actually happening in the nondispersive situation. In this situation, you have a straight line, nondispersed medium actually give you always the group velocity equal to phase velocity.

So that means the carrier and the envelope is going to be moving in the same direction at the same speed. On the other hand, in this case, we can actually have a situation that the phase velocity is actually faster than the group velocity. So what I mean is actually the situation here. The phase velocity calculated from a line connecting from 0 to that point is actually having a larger slope compared to the tangential line. And you see this situation. So basically, you see that inside the envelope all those carriers are actually moving faster than the envelope.

Now I can have a dispersive medium where the group velocity is equal to 0. So what is going to happen is that really the envelope is actually not moving. It's not like this. The body is not moving. So you have some carriers inside this structure is actually moving forward. But the envelope is actually not moving.

So, finally the last situation is really interesting. So in this situation, this is actually having the group velocity-- the group velocity is actually having difference sine compared to the phase velocity. So you can see that the whole structure of the envelope is actually moving backwards. But the carrier is actually moving in the positive direction in this example.

So this is actually what we have learned from this beat phenomenon, and then we have covered the idea of phase velocity and the group velocity. So how about bound system how do we understand when we have a bound system? And how does that evolve as a function of time? So if I have a system of two walls and one string, and of course, I give you the density for the unit length and the string tension, and also the alpha, which is actually telling you about the stiffness of the system.

Again, I can write down psi xt to be the sum of all the normal mode from one to infinity, A m sine km x plus alpha m sine omega mt plus beta m. And then what we can do is that we can first get the initial conditions of this system, and those are the boundary conditions of this system. That we actually just follow exactly the same procedure to obtain all the unknown coefficients that we would be able to evolve this system as a function of time, as I have demonstrated to you in the beginning of the lecture.

So in this case, you can have two boundary conditions. One is actually say at x equal to 0. And the other one is actually at x equal to L. In those boundaries, as we actually learned before, because the endpoints are fixed on the wall. Therefore, psi of 0 at that time, t, will be always equal to 0 for the left-hand side boundary condition. And very similarly, as we discussed before, psi of L t will be equal to 0 if you look at the right-hand side of the wall-- of the system.

So I don't want to repeat this, because this is actually exactly the same calculation which we have done before. So with these two boundary conditions, we can actually conclude that k m will be equal to m pi over L, and alpha m will be equal to 0. So you can actually go back and check this out. So what I'm going to say is that until now, what we have been doing is identical to what we have been doing for the nondispersive media.

What I'm to say is that the shape of the normal mode is actually set by the boundary condition. It's determined by the boundary condition, and it has actually, so far, nothing to do with the dispersion relation omega as a function of k. So in short, boundary condition can give you the shape of the normal mode, and that we know that the first normal mode, second normal mode, et cetera, et cetera, is actually going to be identical to the case of nondispersive medium. so that's actually the first thing which we learned.

The second thing we learned is that OK, now what we see is that once the boundary condition is given, then the k m is actually also given. Therefore, since I have the dispersion relation omega as a function of k, as shown there. Omega over k is equal to v times square root of 1 plus alpha k squared. Therefore, once k m is given, omega m is also given. So you can see that that's actually where the dispersion relation come into play.

The omega m will be different if you compare the dispersive case and nondispersive case. So that is actually what I want to say. The k m, which is the shape of the normal mode, doesn't depend on the dispersion relation. On the other hand, the speed of the oscillation, the angular frequency, omega, depends on the dispersion relation, which is actually what we obtained from there.

If I start to plot omega m as a function of k m-- so in the case of nondispersive medium, so what am I going to get is actually discrete points along a straight line. This is actually k1, k2, k3, k4, et cetera. They are actually all sitting on a common straight line. If you look at the relative difference between k1, k2, and k3, they are constant according to this formula. The difference between k1 and k2 is pi over 2. k2 and k3 is actually also pi over 2-- pi over L. It's always a fixed number. And since omega is actually proportional to k. Therefore, the spacing between omega 1, omega 2, omega 3, is also constant.

In short, omega 2, omega 3, and omega 4, et cetera is always multiple times what you get from omega 1, according to this graph and in the case of nondispersive medium. So what does that mean? That means OK, now if I have a very complicated initial condition-- this is actually what I have, an initial condition-- very complicated.

I just need to wait. If this is actually nondispersive medium, I just have to wait until p equal to 2 pi over omega 1. Then the system would restore to its original shape. That's actually what I can learn from here, because omega 2, omega 3, and any higher order normal modes, the angular frequency is actually multiple times of what I get from omega 1.

On the other hand, if I consider a situation of dispersive medium-- you can see that now the difference between omega m is now the constant. So what you would predict is that it would take much, much longer for this system to go back to the original shape compared to nondispersive media. So that actually you can actually see.

In a real-life experiment, I can distort this equipment in this boundless system, and it's actually going to take forever or impossible to come back to the original shape, because of that dispersion. On the other hand, if I have a really highly idealized situation, if I have both ends bound, and I just have to wait until t equal to 2 pi over omega 1. Then this system will go back to the original shape.

Before I end the lecture today, I would like to discuss with you two interesting issues. So many of you have seen water waves, and Feynman actually told us in his lecture that water waves are really easily seen by everybody, but it's actually the worst possible example. That's the bad news-- the worst possible example because it has all the possible complications that waves can have. That's the bad news. The good news is that you are going to do that in your P set.

[LAUGHTER]

So we will be able to understand the behavior of the water waves. So that's the good news. The second thing which I would like to talk about is phase velocity. You can say, OK, you say that phase velocity or harmonic waves doesn't send information, right? And how do I actually know that? Right? So what does that mean? OK, so let's take this horrible example of water wave. OK, so the black line is actually the beach, and there is a water wave from the ocean approaching the beach. And you can see that you can have some kind of angle between the insert of water wave and the line of the beach.

What I can actually do is that I can now measure the shape of the water wave at the edge of the beach. And I would see that, huh, now the phase velocity which I observe there is actually faster than the speed of propagation of the water wave, because of this inserted angle, OK? I can actually make it very, very fast. I can make the speed actually even faster than the speed of light. Right? I can now decrease the theta to 0. Then you will have a phase velocity which is faster than the speed of light. It goes to infinity.

But does that mean anything? Actually, that doesn't mean anything, because I don't really move the water from a specific point to another point infinitely fast. Therefore, what I want to say is that, OK, you can do whatever you want to make a fancy phase velocity. But that will not help you with sending things close to the speed of light or greater than the speed of light. So as you can see from this example, I can easily construct a simple example, which you see that is actually really not sending anything from one place to the other. But you still have really, really fast phase velocity.

OK, thank you very much, everybody, for the attention and hope you enjoyed the lecture. And if you have any questions, please let me know.