In many media : $v = v(\lambda)$

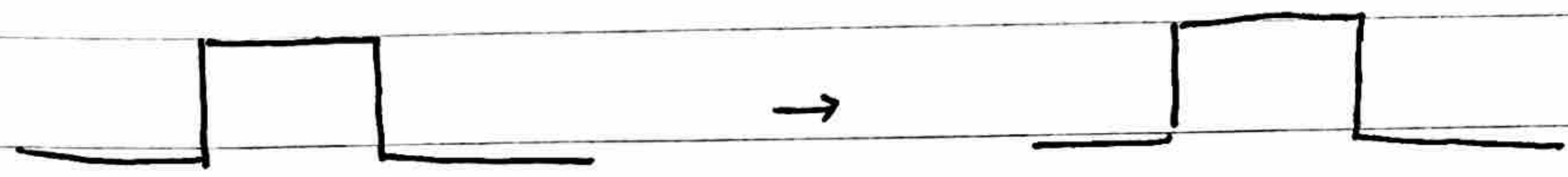
For instance : light in glass

* Speed of wave propagation depends on wavelength λ

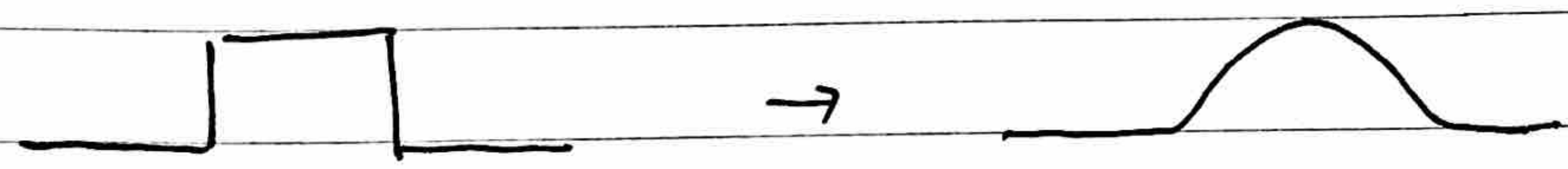
Red > Violet

(or ω or k)

* Deep water * Beaded string * realistic string * ...



Non-dispersive medium



Dispersive medium

made of many different modes → traveling at different speed

→ "disperse"

Phase velocity : $\boxed{v_p = \frac{\omega}{k}}$

Group velocity $\boxed{v_g = \frac{d\omega}{dk}}$

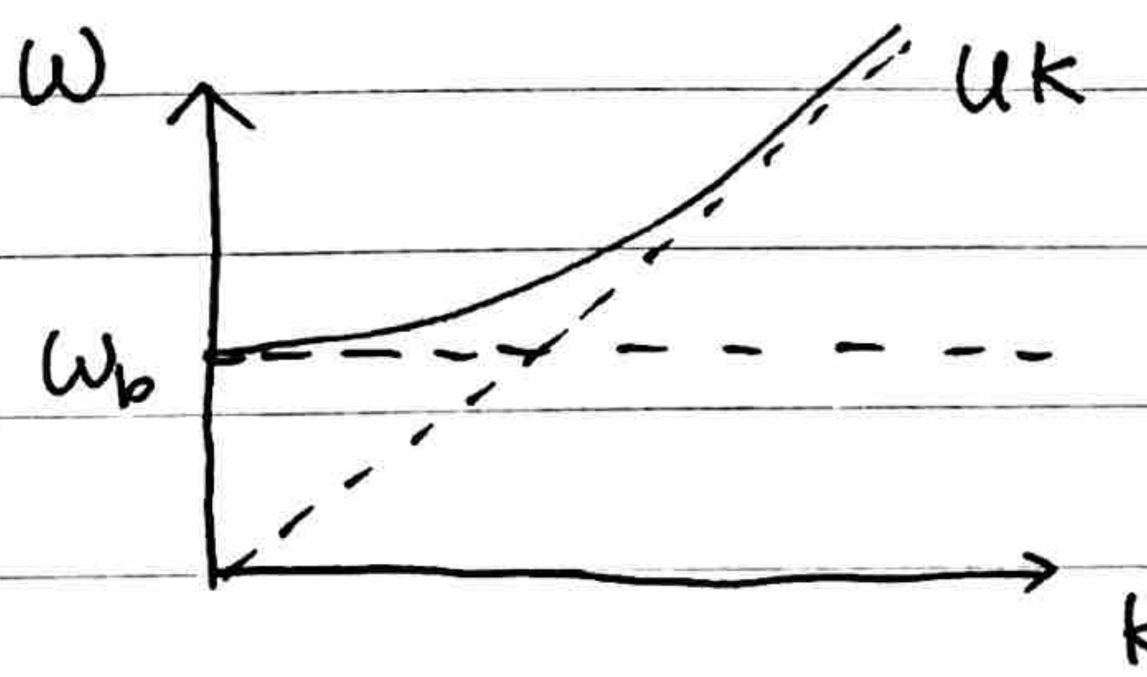
* Non-dispersive medium $\omega = v \cdot k$

$\Rightarrow$ Phase velocity $= v$

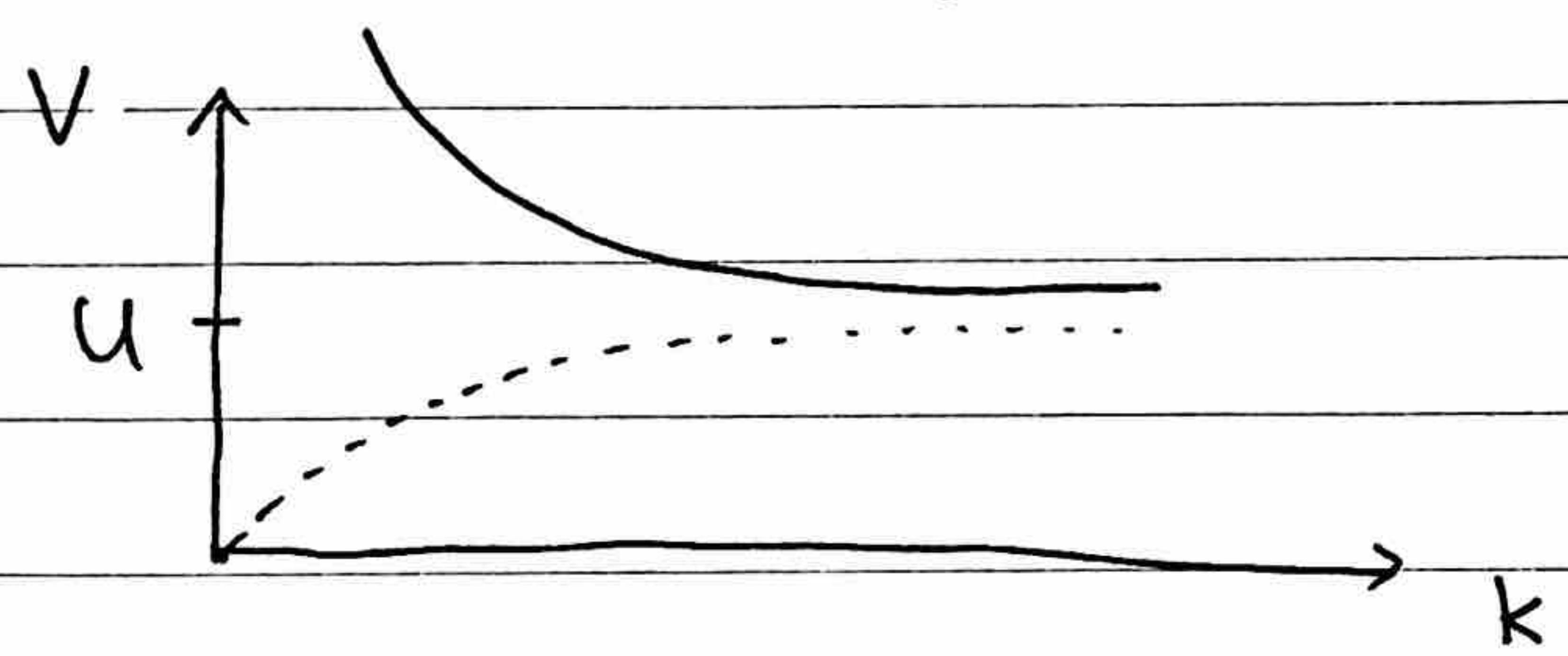
Group velocity $= \frac{d\omega}{dk} = v =$ Phase velocity

Example:

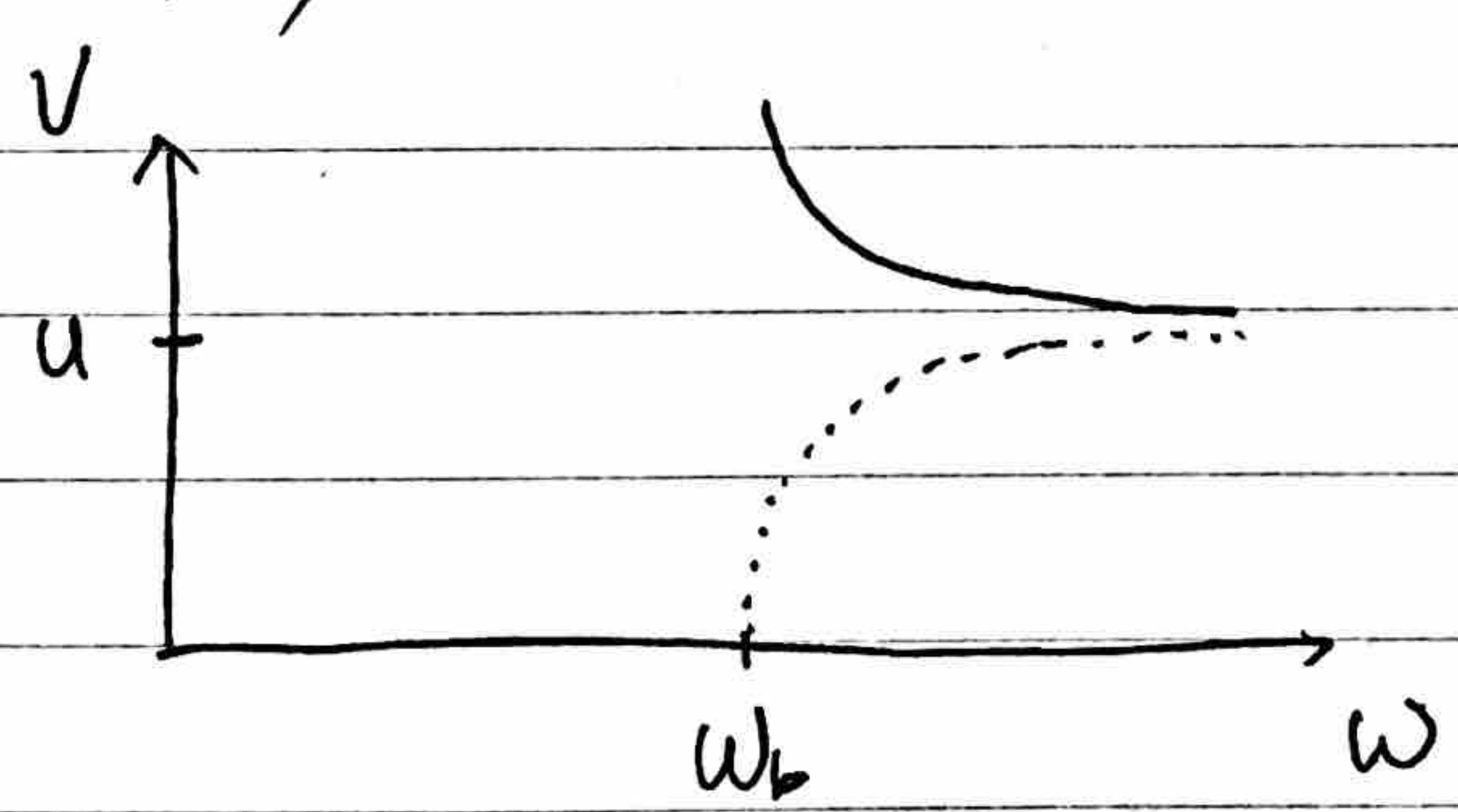
EM wave passing through an ionic crystal
The dispersion curve looks like



(1) What is the group velocity and phase velocity as a function of k ?



(2) Velocity v.s. ω ?



(3) What will happen to radiation striking such a crystal if the frequency is $\omega < \omega_b$?

There is no propagation or loss in this crystal

⇒ totally reflected!

If we have a very long string:

We shake one end. $\Rightarrow$ Produce a progressive wave!

$$\psi(x,t) = f\left(t - \frac{x}{v}\right)$$ for non-dispersive medium.

How about dispersive medium?

$\Rightarrow$ Waves with different frequency (or wave length) are traveling at different speed!

$\Rightarrow$ Need to decompose $f(t)$ into waves with fix frequency

Then attack them one by one!!!

Use Fourier transform this mathematical tool!

$$f(t) = \int_{-\infty}^{\infty} d\omega \; \underset{\Uparrow}{C(\omega)} \; \underset{\Uparrow}{e^{-i\omega t}}$$

Amplitude   oscillation at ω

After we decompose $f(t)$ into many harmonic oscillations with different frequency and use the dispersion relation $\omega = \omega(k)$

$$\psi(x,t) = \int_{-\infty}^{\infty} d\omega \; C(\omega)\, e^{-i\omega t + i k(\omega) x}$$

With a given ω we can solve k (a function of ω)

Special Case : Non-dispersive system :

$$K(\omega) = \frac{\omega}{v}$$

$$\psi(x,t) = \int_{-\infty}^{\infty} d\omega \; C(\omega)\, e^{-i(\omega t - \frac{\omega}{v} x)}$$

$$= \int_{-\infty}^{\infty} d\omega \; C(\omega)\, e^{-i\omega(\underline{t - \frac{x}{v}})}$$

$$= f(\underline{t - \frac{x}{v}})$$

Make sense!

How do we determine $C(\omega)$?

Orthogonality :

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\omega-\omega')t}\,dt = \delta(\omega-\omega')$$

$$\delta(x) = \begin{cases} \infty & x=0 \\ 0 & x\neq 0 \end{cases}$$

Dirac Delta function !

$$\delta_n(\omega) = \frac{1}{2\pi}\int_{-n}^{n} e^{i\omega t}\,dt$$

$$= \frac{1}{2\pi i\omega}\left(e^{in\omega t} - e^{-in\omega t}\right)$$

$$= \frac{\sin nx}{\pi x}$$

$$\int_{-\infty}^{\infty} \delta_n(x)\,dx = 1$$

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} \delta_n(x) f(x)\,dx = f(0)$$

Some useful formula :

$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1$$

$$\int_{-\infty}^{\infty} \delta(x-\alpha) f(\alpha)\,d\alpha = f(x)$$

Now if I calculate this quantity :

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, f(t)\, e^{i(\omega t)}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dt \left(\int_{-\infty}^{\infty} C(\omega')\, e^{-i\omega' t}\, d\omega'\right) e^{i\omega t}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} C(\omega')\,d\omega' \int_{-\infty}^{\infty} dt\; e^{i(\omega-\omega')t}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} C(\omega')\,\delta(\omega-\omega')\,d\omega' = \underline{C(\omega)}$$

Now we have a problem :

Not good.

How do we overcome this difficulty ?

A smart idea : AM radio

$f_s(t)$: the signal we want to transmit.

For instance : music ; sound $\sim$ 1 KHz

Carrier : $\cos \omega_0 t$ or $e^{i\omega_0 t}$

The frequency of the AM radio $\sim$ 0.1 - 30 MHz

Instead of transmiting $f_s(t)$ directly (we know this doesn't work) We transmit

$$f(t) = f_s(t) \cos \omega_0 t$$

DEMO

Since $f_s(t)$ is SLOW ($\sim$ 1 KHz) compared to $\cos \omega_0 t$ ($\sim$ 0.1 $\sim$ 30 MHz)

$\Rightarrow$ the resulting ω range with non-zero $C(\omega)$ is "narrow"

This is because:

$$\cos \omega_s t \ \cos \omega_0 t$$
$$= \frac{1}{2}\left[\cos(\omega_0+\omega_s)t + \cos(\omega_0-\omega_s)t\right]$$

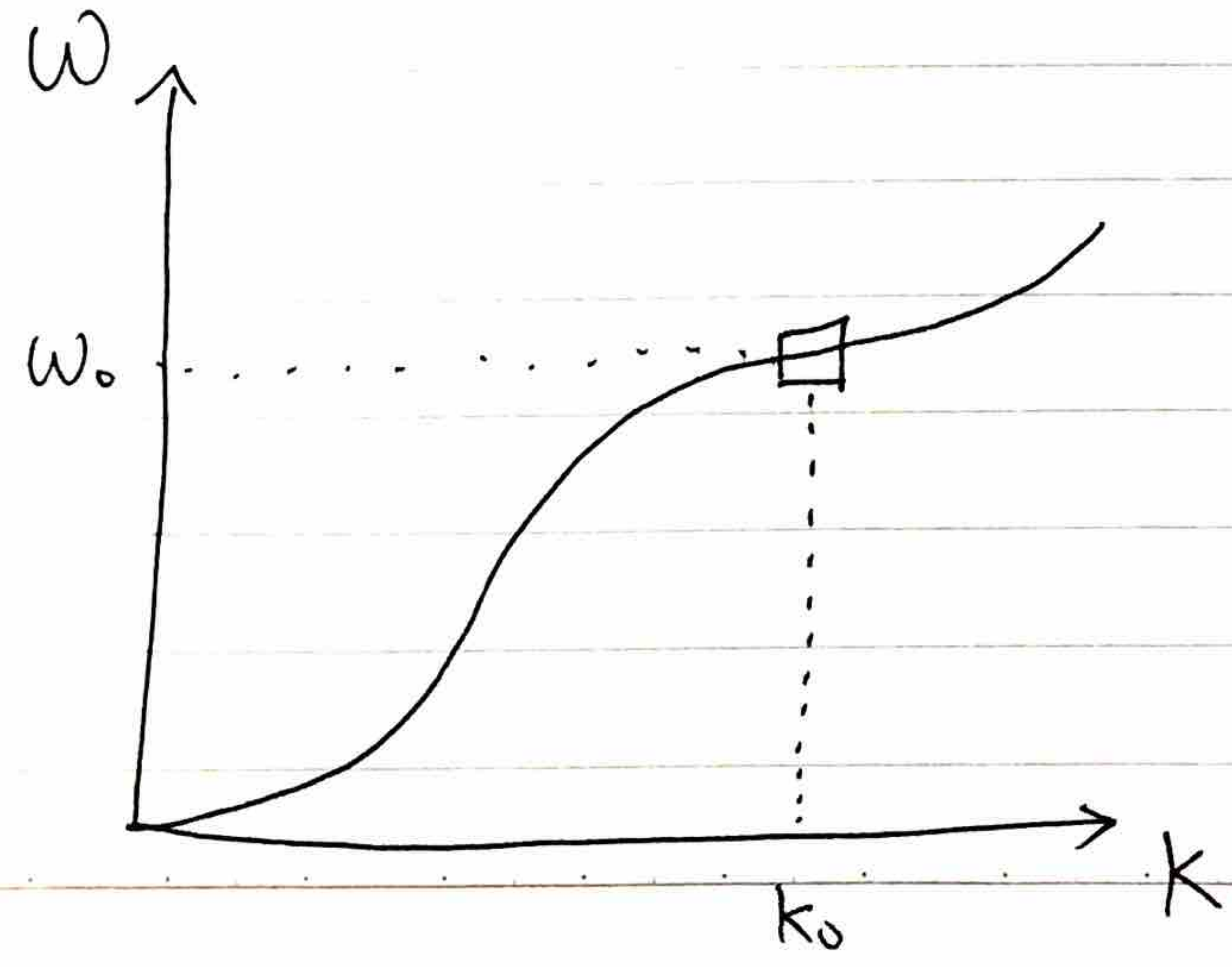
ω_s is the "typical frequency" of the signal

ω_0 is the carrier frequency.

Therefore, the range of ω with non-zero $C(\omega)$ is $\approx$ $\omega_0-\omega_s$ to $\omega_0+\omega_s$

Where ω_s is $\ll \omega_0$.

Dispersion Relation: $\omega = \omega(k)$



Suppose $\omega(k)$ is slowly varying around ω_0

(Taylor Expension)

$$\omega = \omega(k) = \omega_0 + (k-k_0)\left.\frac{\partial \omega}{\partial k}\right|_{k=k_0} + \cdots$$

$\parallel$

v_g (a)

$$\Rightarrow \boxed{\omega \approx \omega_0 + (k-k_0)\, v_g}$$

$\omega_0 \equiv \omega(k_0)$ when $k \approx k_0$, $\omega \approx \omega_0$

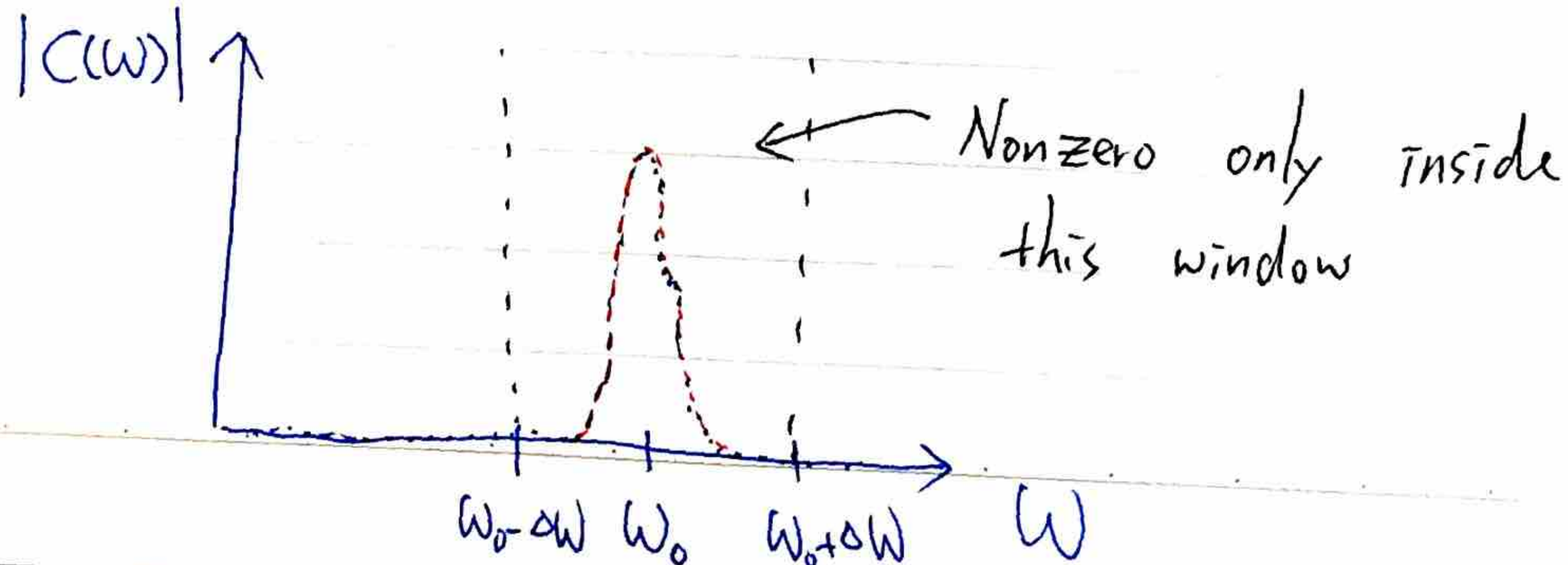
$\Rightarrow$ Higher order terms are negligible in the range

$$\omega_0 - \Delta\omega < \omega < \omega_0 + \Delta\omega$$

If my $f(t)$ satisfies with $C(\omega) \approx 0$

for $|\omega - \omega_0| > \Delta\omega$

ie. we are looking at $f(t)$ with the corresponding $C(\omega)$ like



$f_s(t)$ must be "slowly-varying" compared to the carrier wave $e^{-i\omega_0 t}$

$$f(t) = \mathrm{Re}(\underbrace{f_s(t)}_{\text{"envelope"}} \underbrace{e^{-i\omega_0 t}}_{\text{"carrier"}})$$

This is actually "Amplitude Modulation"

AM radio !

If $\Delta\omega \ll \omega_0$ (a small window with $C(\omega) \neq 0$)

$\Rightarrow$ higher order terms in $\omega(k)$ doesn't matter or negligible

from p10. (a)

$$\Rightarrow \quad \omega = v_g k + a \qquad a = \omega_0 - v_g k_0$$

$$k = \omega / v_g + b \qquad b = k_0 - \omega_0 / v_g$$

a and b are constants.

Now we want to show

$$\psi(x,t) = \mathrm{Re}\left(f_s\left(t - \frac{x}{v_g}\right) e^{-i(\omega_0 t - k_0 x)}\right)$$

Fourier transform : we can rewrite $f_s(t)$ as

$$f_s(t) = \int_{-\infty}^{\infty} dw \; c(w) \, e^{-iwt}$$

Make AM radio, multiply by $e^{-iw_0 t}$

$$\underset{f_s(t)\,e^{-iw_0 t}}{\underset{\shortparallel}{f(t)}} = \int_{-\infty}^{\infty} dw \; c(w) \, e^{-i(w+w_0)t}$$

$$= \int_{-\infty}^{\infty} dw \; c(w-w_0) \, e^{-iwt}$$

Propagate to all x

$$\psi(x,t) = Re\left[\int_{-\infty}^{\infty} dw \; \underset{\Downarrow\ (b)}{c(w-w_0)} \, e^{-iwt} \, e^{ikx}\right]$$

$C(w)$ is only nonzero around w_0

$$(b) \quad \approx \int_{-\infty}^{\infty} dw \; c(w-w_0) \, e^{-iwt} \, e^{i(w/v_g + b)x}$$

$$= \int_{-\infty}^{\infty} dw \; c(w-w_0) \, e^{-iw(t-\frac{x}{v_g})} \, e^{ibx}$$

$$= \int_{-\infty}^{\infty} dw \; c(w) \, e^{-i(w+w_0)(t-\frac{x}{v_g})} \, e^{ibx}$$

rewrite

$$(b) \quad = \underbrace{\int_{-\infty}^{\infty} dw \; c(w) e^{-iw(t-\frac{x}{v_g})}}_{f_s(t-\frac{x}{v_g})} \, e^{-iw_0 t} \, \underbrace{e^{i(\frac{w_0}{v_g}+b)x}}_{e^{ik_0 x}}$$

Therefore

$$\psi(x,t) = Re\left[f_s\left(t-\frac{x}{v_g}\right) e^{-i(w_0 t - k_0 x)}\right]$$

$$\Rightarrow \psi(x,t) = \mathrm{Re}\Big[\underbrace{f_s\left(t-\frac{x}{v_g}\right)}\ \underbrace{e^{-i(\omega t - k_0 x)}}\Big]$$

$v_p = \dfrac{\omega_0}{k_0}$

The envelope traveling at v_g — Group velocity!

The Carrier traveling at v_p — Phase Velocity!

What is the typical carrier frequency?

Medium Frequency 300 kHz ↔ 3 MHz

↳ Skywave

High Frequency 3 MHz ↔ 30 MHz

$f_s(t)$

The envelope shape doesn't change !!

(No dispersion)

Enable us to send voice, music to places which are thousands of miles away!!!