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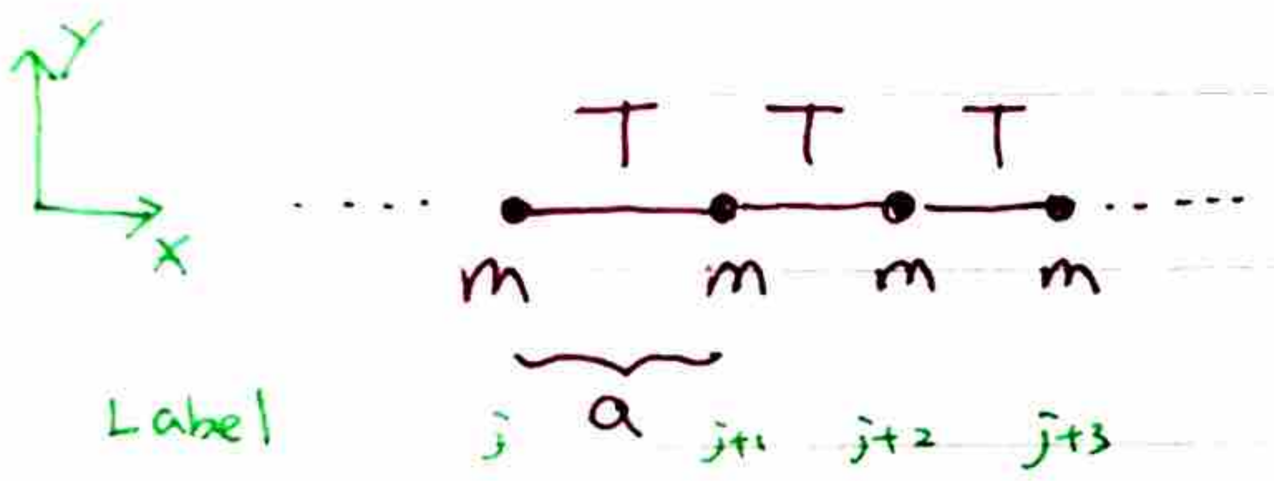
Review:

We have discussed about this system in lecture 8



infinitely long !

Mass can only move up and down $\hat{y}$ direction

We have solved it by "space translation symmetry"

Dispersion relation $\omega(k)$ we obtained was:

$$\omega(k) = \frac{T}{ma} \sin \frac{ka}{2}$$

T: String tension a: distance between masses (at equilibrium position)

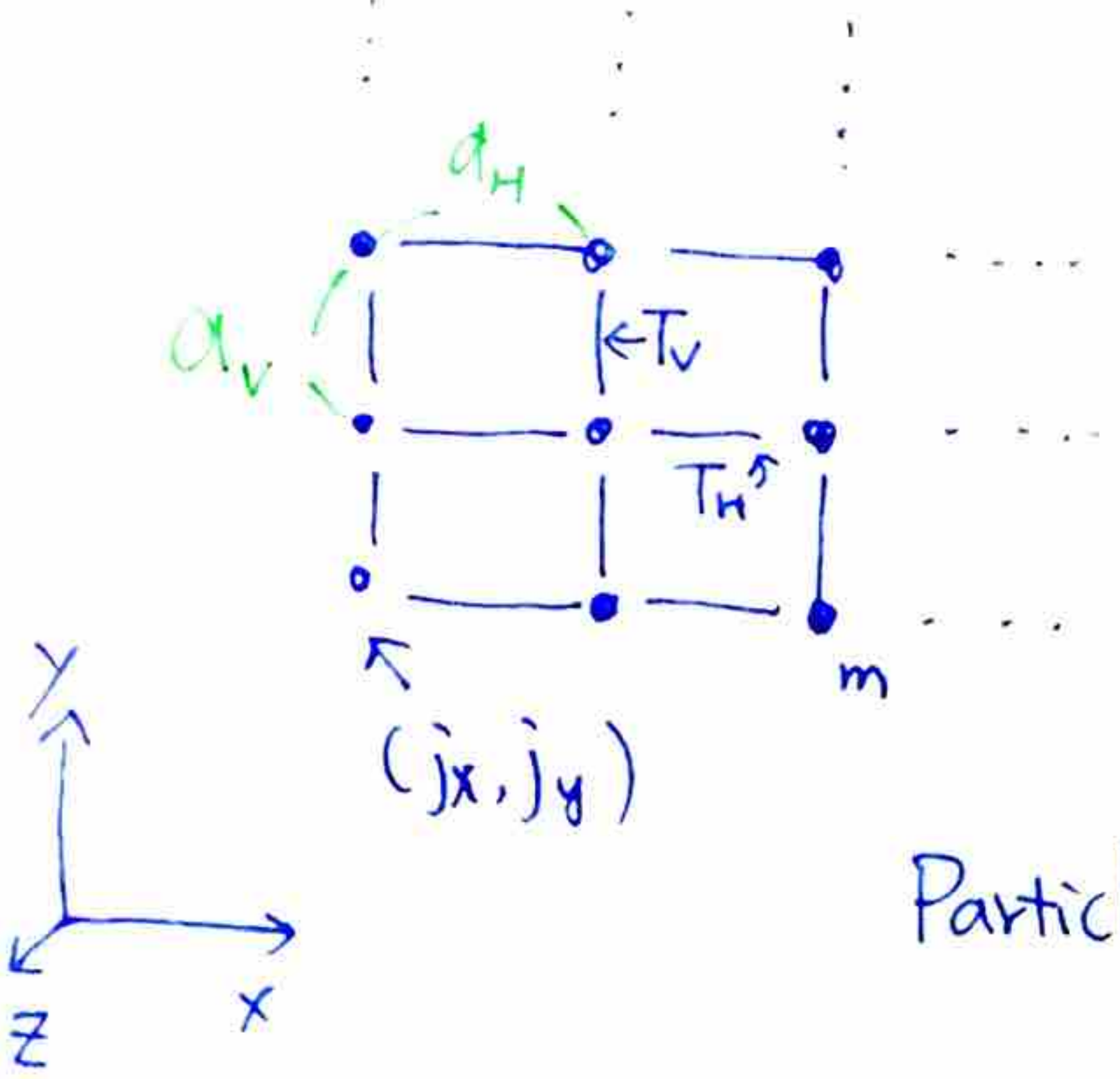
m: mass eigenvectors: e^{ikx}, $x = \underbrace{j}_{\text{Label}} \cdot a$

Today: 2D / 3D system !!

In general, we don't know how to solve those systems ! :(

But we know how to solve highly symmetric systems !!

If we consider an infinitely long array of masses



mass m

Tension T_V , T_H

Ideal string

Particles can only move in the $\hat{z}$ - direction

Good news : Space translation symmetry ! ☺

$\Rightarrow$ Eigenvector : $e^{ik_x x} e^{ik_y y}$

where $\begin{cases} x = \underbrace{j_x}_{\text{Label for } \hat{x}} a_H \\ y = \underbrace{j_y}_{\text{Label for } \hat{y} \text{ direction}} a_V \end{cases}$ (j_x , j_y) index

$$\Rightarrow \Psi(x,y) = A e^{ik_x x} e^{ik_y y} = A e^{i\vec{k}\cdot\vec{r}}$$

We can use the expression above to get the dispersion relation

$$\omega^2 = \frac{4T_H}{m a_H} \sin^2 \frac{k_x a_H}{2} + \frac{4T_V}{m a_V} \sin^2 \frac{k_y a_V}{2}$$

by the way, this is a dispersive medium.

$\because \dfrac{\omega}{|\vec{k}|}$ is not a constant !

At fixed ω :

If we consider 1D bead - string system :

There are two solutions (or eigenvectors of S matrix) which gives angular frequency ω

$$e^{ikx} \quad \text{and} \quad e^{-ikx}$$

The idea we got was :

Nice ! This is $\cos kx$ and $\sin kx$!!

$$\cos kx = \frac{1}{2}\left(e^{ikx} + e^{-ikx}\right)$$

$$\sin kx = \frac{1}{2i}\left(e^{ikx} - e^{-ikx}\right)$$

or $\cos(kx+\phi)$

We know from the discussion above, the

eigenvector of $M^{-1}K$ matrix is sin or cos.

Back to two-dimensional case :

If we fix the angular frequency to be ω

There are multiple values of k_x and k_y which can give the same ω

(actually infinite # of choice)

This is because k_x and k_y are continuous ; can be any value before we introduce boundary condition

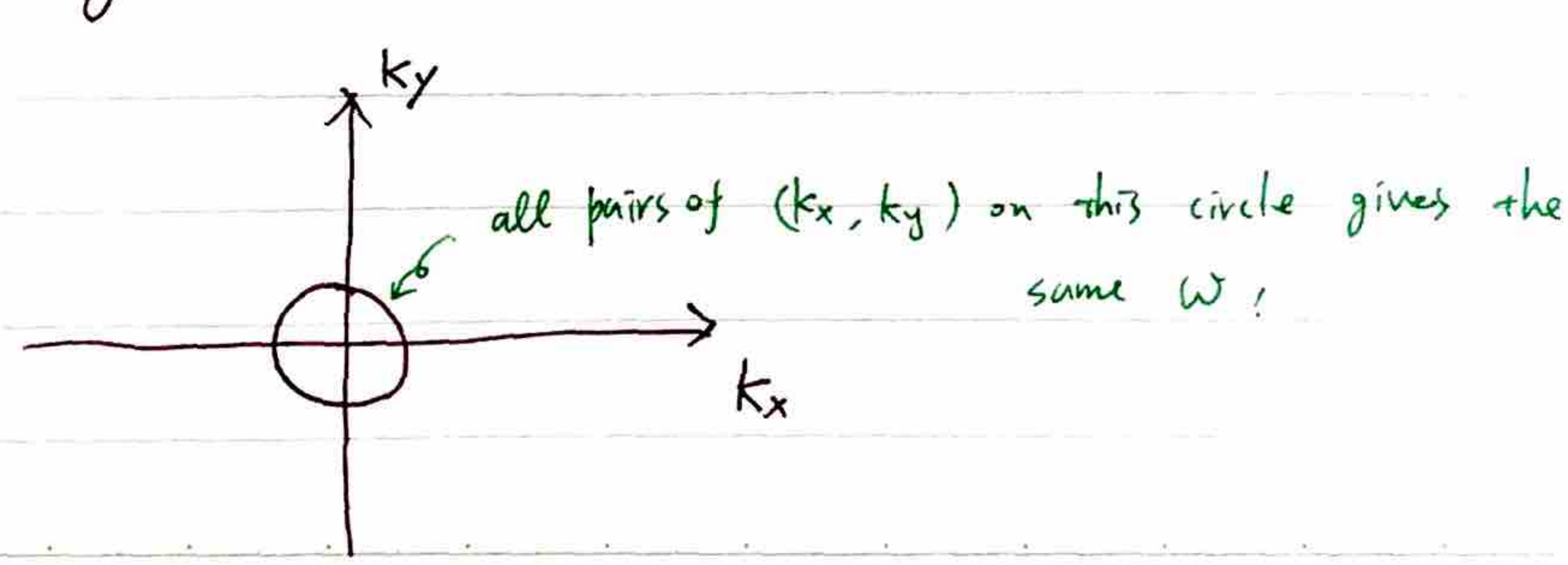
If we lower k_x a bit , we can increase k_y to compensate that !

Example : If I have dispersion relation in this form :

$$\omega^2 = 5 \sin^2 k_x + 5 \sin^2 k_y$$

(slides)

There are many possible pairs of k_x and k_y which gives the same ω !!!



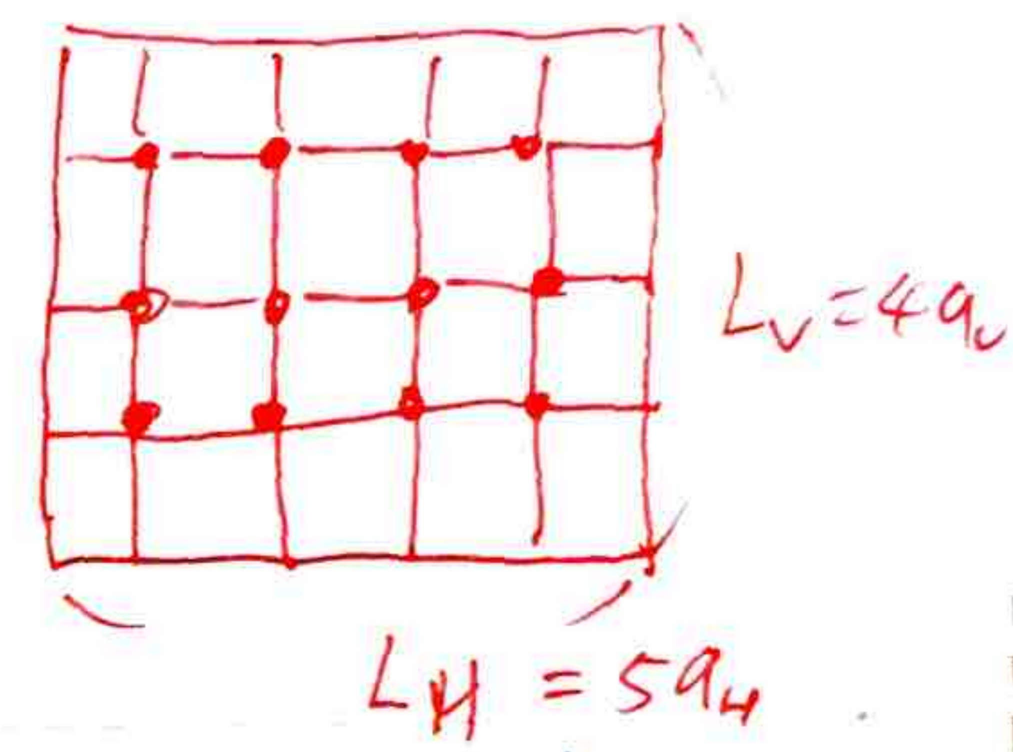
Now we add the wall back in:

$\boxed{\text{B.C.}}$

$$\psi(0, y, t) = \psi(L_H, y, t)$$

$$= \psi(x, 0, t) = \psi(x, L_v, t) = 0$$

In this example $\begin{cases} L_H = 5a_H \\ L_V = 4a_V \end{cases}$



There are now only four modes of the finite system with the same ω

$$A e^{\pm i k_x X} e^{\pm i k_y Y}$$

$$k_x = \frac{n_x \pi}{L_H} \qquad k_y = \frac{n_y \pi}{L_V}$$

$$L_H = 5a_H \qquad L_V = 4a_V$$

$n_x = 1 \sim 4$
$n_y = 1 \sim 3$

$\Rightarrow$ Linear combination of $e^{+ik_x X} e^{+ik_y Y}$, $e^{+ik_x X} e^{-ik_y Y}$

$e^{-ik_x X} e^{+ik_y Y}$, $e^{-ik_x X} e^{-ik_y Y}$

$\Rightarrow$ gives $A \sin k_x X \sin k_y Y$ which satisfy the boundary conditions above.

Write down the $(n_{x\,th}, n_{y\,th})$ normal mode:

obtain from dispersion relation

$$\sin(\omega_{(n_x,n_y)} t + \beta_{(n_x,n_y)})$$

$$\Rightarrow \psi_{(n_x,n_y)}(x,y,t) = A_{(n_x,n_y)} \sin \frac{n_x \pi X}{L_H} \sin \frac{n_y \pi Y}{L_V}$$

Discrete Case:

General solution $\psi(x,y,t) = \sum_{n_x,n_y} A_{(n_x,n_y)} \sin\frac{n_x\pi x}{L_H} \sin\frac{n_y \pi y}{L_V} \cdot \sin(\omega_{(n_x,n_y)} t + \beta_{(n_x+n_y)})$

Continuous case:

Assuming $T_H = T_V = T$

$a_H = a_V = a \longrightarrow 0$

$$\Rightarrow \omega^2 = \frac{4T}{ma}\frac{k_x^2 a^2}{4} + \frac{4T}{ma}\frac{k_y^2 a^2}{4}$$

$$= \frac{Ta}{m}(k_x^2 + k_y^2)$$

Define $\rho_s = \frac{m}{a^2}$ surface mass density $\quad T_s = \frac{T}{a}$ surface tension

$$\Rightarrow \omega^2 = \frac{T_s}{\rho_s}(k_x^2 + k_y^2) = \frac{T_s}{\rho_s}|\vec{k}|^2$$

Similar to 1-D case. Continuous limit gives

$$\frac{\partial^2}{\partial t^2}\psi(x,y,t) = v^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi(x,y,t)$$

$$= v^2 \nabla^2 \psi(x,y,t)$$

$$v = \sqrt{\frac{T_s}{\rho_s}}$$

$$\Rightarrow \frac{\partial^2}{\partial t^2}\psi(x,y,t) = v^2 \nabla^2 \psi(x,y,t)$$

$\psi \propto A \sin k_x x \sin k_y y \sin(\omega t + \phi)$

DEMO

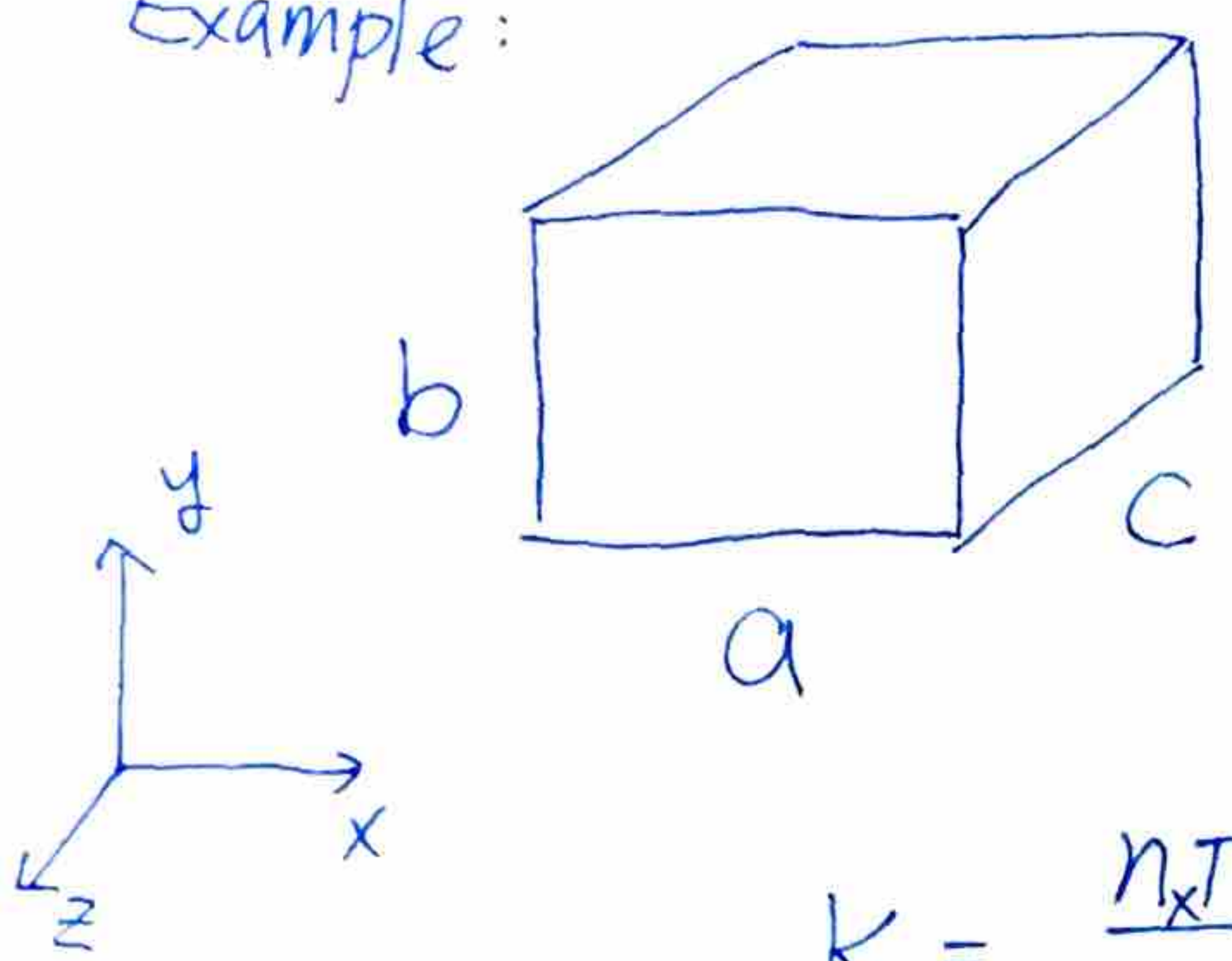
Similarly 3D case:

$$\frac{\partial^2}{\partial t^2}\psi(x,y,z,t) = v^2 \nabla^2 \psi(x,y,z,t)$$

Continuous Case:

3D sound wave. $\nabla^2 \psi = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}$

Example: Sound wave in a box



$$K_x = \frac{n_x \pi}{a} \qquad K_y = \frac{n_y \pi}{b} \qquad K_z = \frac{n_z \pi}{c}$$

Guess $\Rightarrow \vec{\psi} \propto \sin(K_x x)\sin(K_y y)\sin(K_z z)\sin(\omega t + \phi)$

Plug into wave equation

$$\Rightarrow \omega^2 = v^2 (K_x^2 + K_y^2 + K_z^2)$$

$$= v^2 \left(\left(\frac{n_x \pi}{a}\right)^2 + \left(\frac{n_y \pi}{b}\right)^2 + \left(\frac{n_z \pi}{c}\right)^2 \right)$$

where n_x, n_y, n_z are integers.

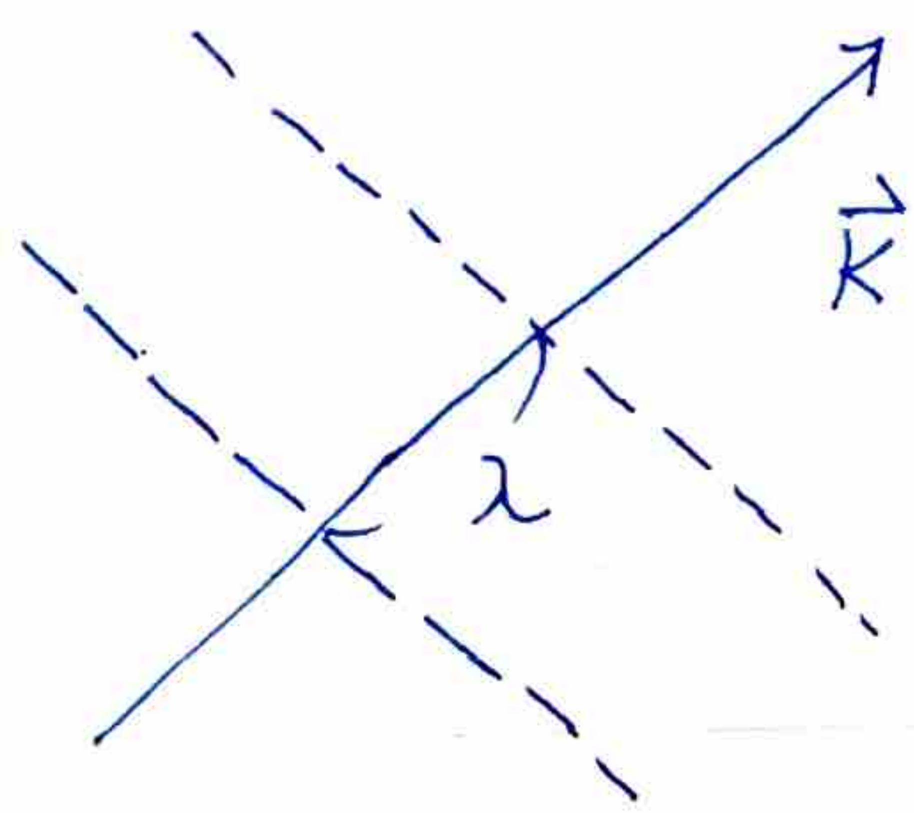
Break?

5/9

2/3D progressive wave :

Simple example : "plane waves"

$$\psi(\vec{r},t) = A\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$



$\vec{k}$: direction of propagation
"wave number vector"

$$\lambda = \frac{2\pi}{|\vec{k}|}$$

This can be used to describe EM waves
Sound waves , or waves on membranes.

If there is no other medium

→ This wave will continue forever.

Let's consider a 2-D membranes stretched in the $z=0$ plane. with surface mass density ρ_s and surface tension T_s

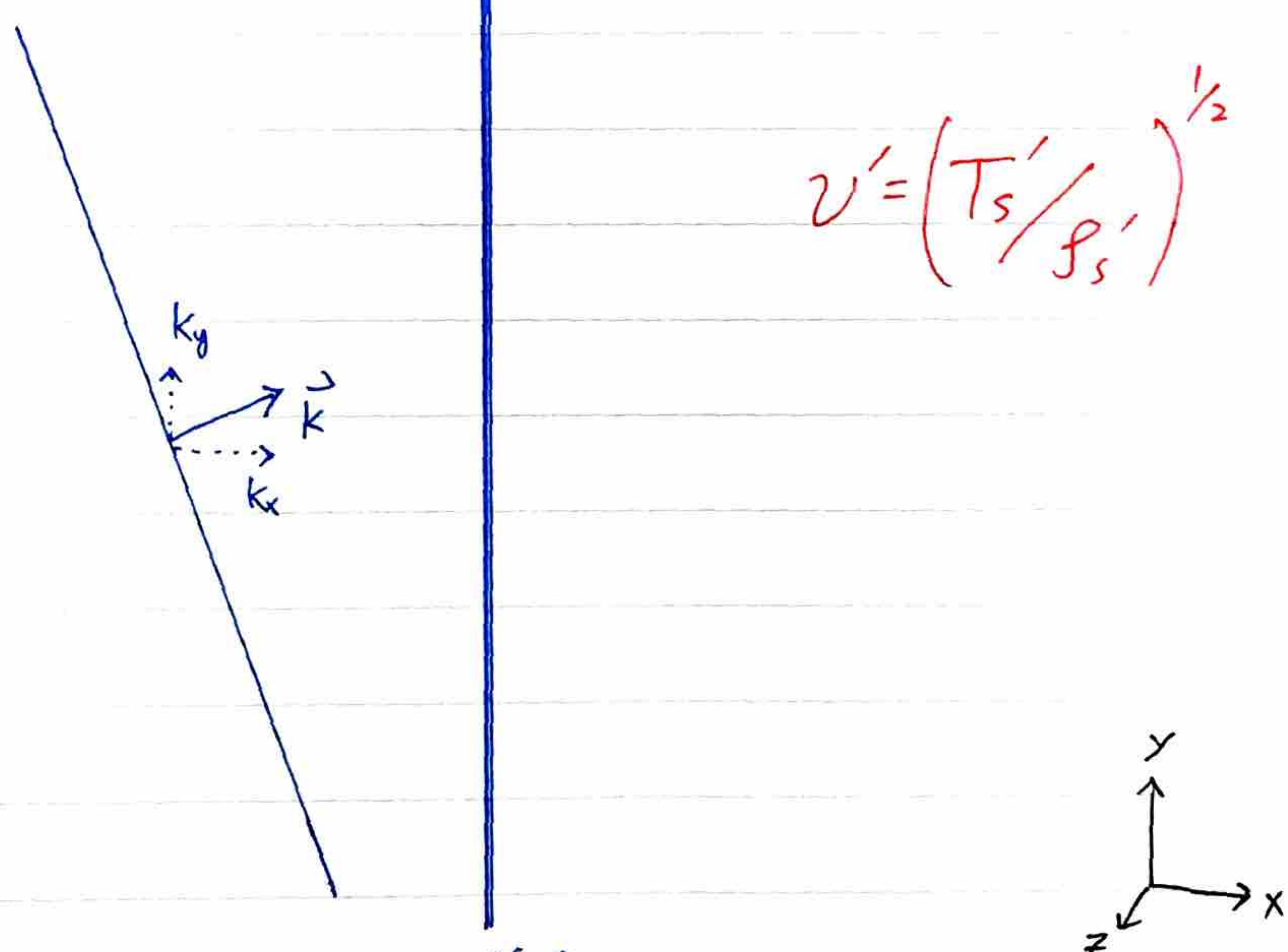
$$\omega^2 = v^2 (k_x^2 + k_y^2)$$

$\Rightarrow$ Waves will travel at a speed of $v = \sqrt{\frac{T_s}{\rho_s}}$

To add some excitement to this problem:

T_s , ρ_s $\qquad$ T_s' , ρ_s'

$$v' = \left(T_s' / \rho_s'\right)^{1/2}$$



Question: What will happen ?

One expect as usual incident wave. reflected and transmitted wave.

Summing over all possible $\vec{k}$.

at the same frequency

$$\psi_L = A e^{i(\vec{k}\cdot\vec{r}-\omega t)} + \sum_\alpha R_\alpha A e^{i(\vec{K}_\alpha\cdot\vec{r}-\omega t)} \qquad (x \leq 0)$$

(Incident wave) (Reflected wave)

$$\psi_R = \sum_\beta T_\beta A e^{i(\vec{K}_\beta\cdot\vec{r}-\omega t)} \qquad (x \geq 0)$$

(Transmit wave)

$$|\vec{K}_\alpha|^2 = \omega^2 \frac{\rho_s}{T_s} = \omega^2/v^2, \quad |\vec{K}_\beta|^2 = \omega^2 \frac{\rho_s'}{T_s'} = \omega^2/v'^2 \quad ①$$

Σ_α, Σ_β : sum over all possible $\vec{K}_\alpha$, $\vec{K}_\beta$ vectors which gives angular frequency ω

To calalate R_α and T_β as well as $\vec{K}_\alpha$, $\vec{K}_\beta$

$\Rightarrow$ Need boundary condition!

Boundary Condition:

$\psi_L = \psi_R$

At $x=0$ and $z=0$ Membrane doesn't break!

$$\psi(0,y,0,t) = A e^{i(k_y y-\omega t)} + \sum_\alpha R_\alpha A e^{i(K_{\alpha y} y-\omega t)}$$

$$= \sum_\beta T_\beta A e^{i(K_{\beta y} y-\omega t)}$$

boundary condition

Only when $K_{\alpha y} = K_{\beta y} = K_y$ ②

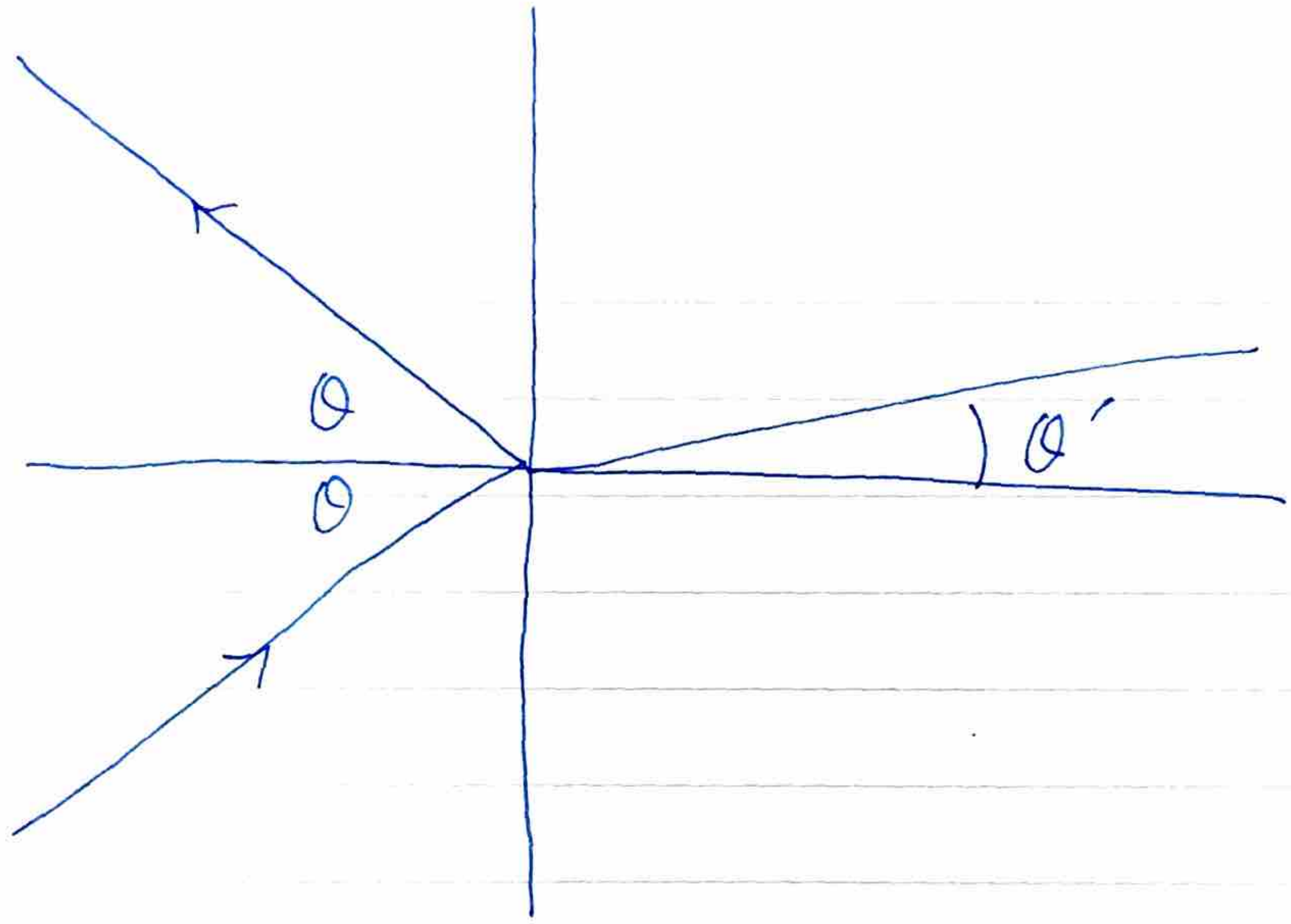
The equality holds!

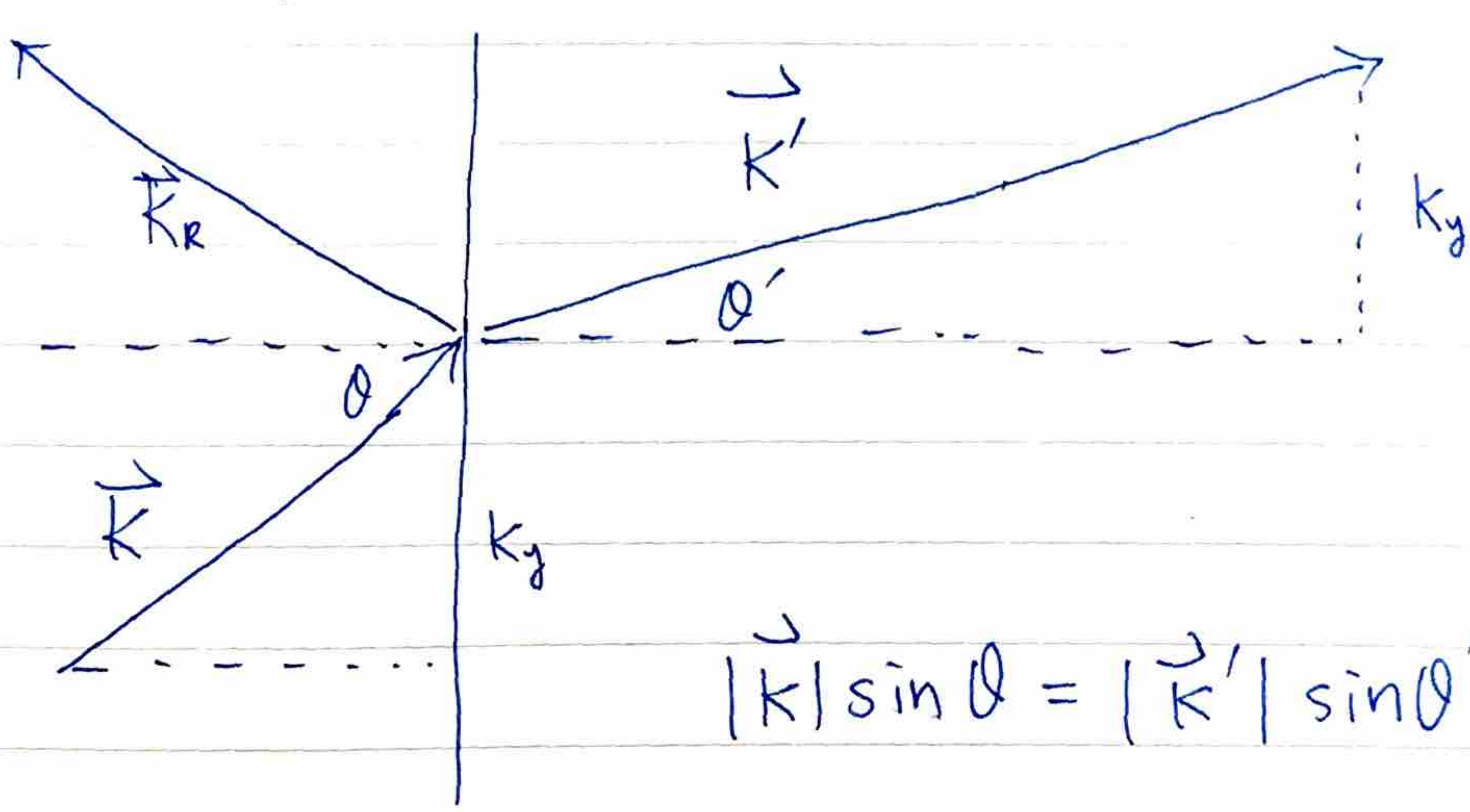
① and ② fixed $\vec{K}_\alpha$ and $\vec{K}_\beta$

$\Rightarrow$ Only when $K_{\alpha x} = -\sqrt{\omega^2/v^2 - K_y^2} = -K_x$

and $K_{\beta x} = \sqrt{\omega^2/v'^2 - K_y^2}$

We can satisfy the boundary condition





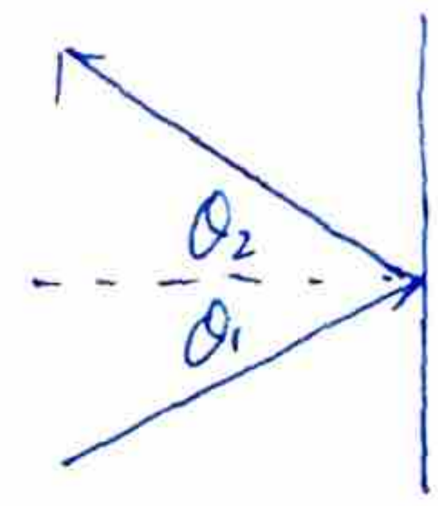
$$|\vec{K}| \sin\theta = |\vec{K}'| \sin\theta'$$

Define $n = \frac{c}{v} = \frac{c}{\omega}|K|$ $\Rightarrow$ $n\sin\theta = n'\sin\theta'$

$n' = \frac{c}{v'} = \frac{c}{\omega}|K'|$ Snell's Law !!!

We have just proved two MOST IMPORTANT LAWS of geometrical optics !!!

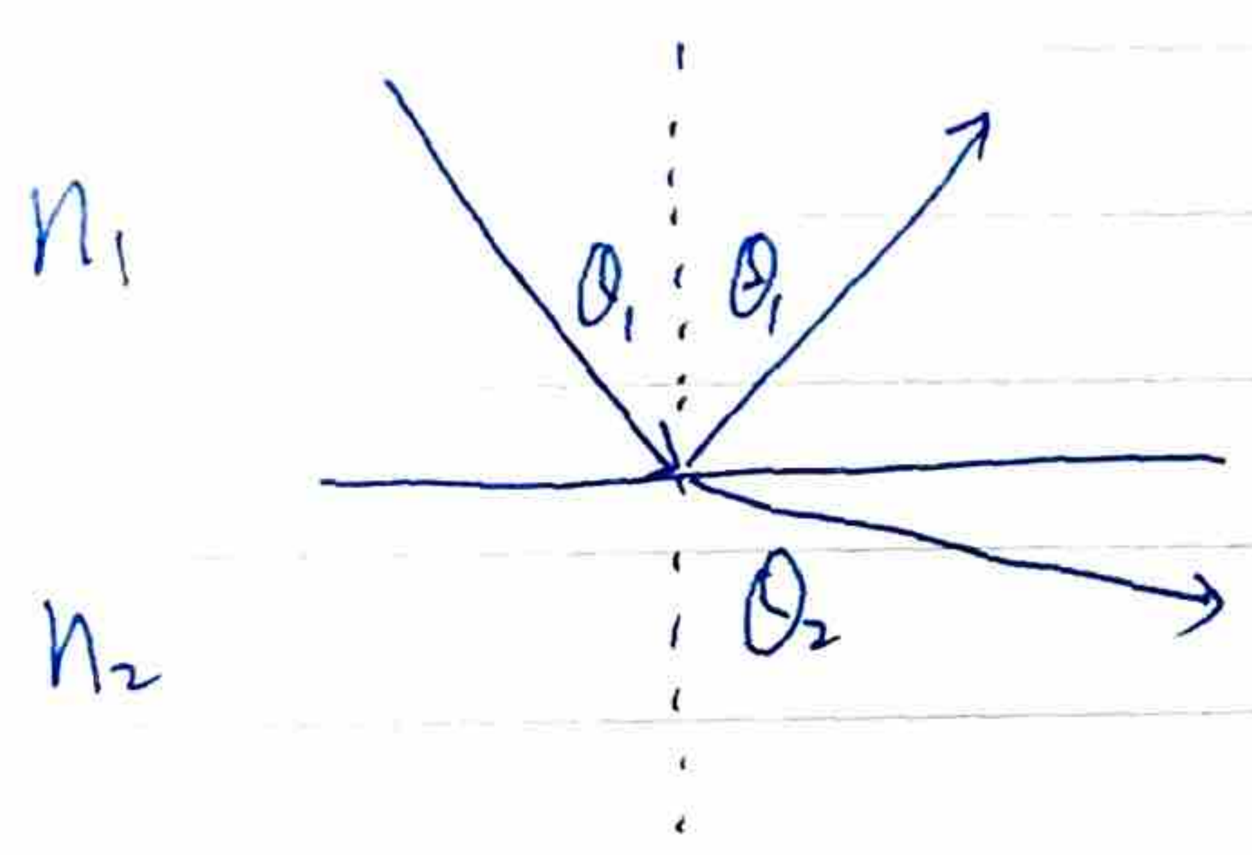
(1) Reflection



$\theta_1 = \theta_2$

(2) Snell's law !

$n_1 \sin\theta_1 = n_2 \sin\theta_2$ $\qquad$ n_1 = refraction index.



if $n_2 < n_1$

(3) It works for water, glass, sound, light waves!

(4) if we continue to increase θ_1

$\Rightarrow \quad \dfrac{n_1}{n_2} \sin\theta_1 > 1$

$\Rightarrow$ No transmission! $\quad$ DEMO

Next time: Polarization! $\quad$ slide