Massachusetts Institute of Technology

Physics 8.03

Practice Exam 2

Problem 1

The charge density fluctuations in a plasma satisfy the wave equation, which for one dimensional distortions reduces to

$$\frac{\partial^2 \rho}{\partial t^2} = c^2 \frac{\partial^2 \rho}{\partial x^2} - \omega_p^2 \rho$$

where $\rho(x, t)$ is the charge density fluctuation, c is the speed of light, and fixed parameter ω_p is known as the plasma frequency.

a. Find the dispersion relation $\omega(k)$ for traveling wave solutions of the form $\rho(x, t) = a \sin(kx - \omega t)$.

b. Find $k(\omega)$. Graph wavevector k as a function of wave frequency ω. Find and sketch phase and group velocities.

c. Show that the wave equation can also be satisfied by exponentially decaying solution of the form:

$$\rho(x, t) = a \cos(\omega t) e^{-\kappa x}$$

Find $\kappa(\omega)$ and graph it (you may use the same graph as in previous question but make sure to label it clearly).

Problem 2

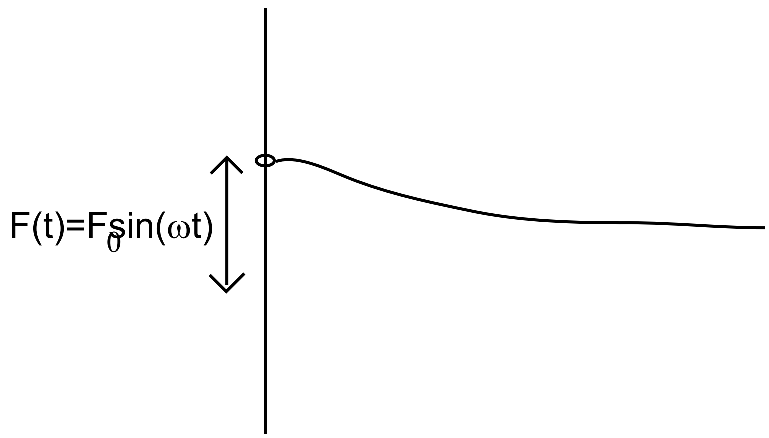

Figure 1: Force creating a wave

Consider a very long string attached to a small massless loop at $x = 0$. The string has mass density μ and it is kept at tension T. The loop can move vertically without friction. An external force $F(t) = F_0 \sin(\omega t)$ is applied to the loop moving it up and down resulting in a steady state harmonic wave traveling towards positive x. Assume that the string is so long that you don't need to worry about the reflected pulses.

a. Find the boundary conditions that need to be satisfied by a wave solution $y(x, t)$ at $x = 0$ in terms of given parameters.

b. Find the frequency and wavelength of the resultant steady-state harmonic wave as a function of given parameters.

c. Find the amplitude of the resultant steady-state harmonic wave.

Problem 3

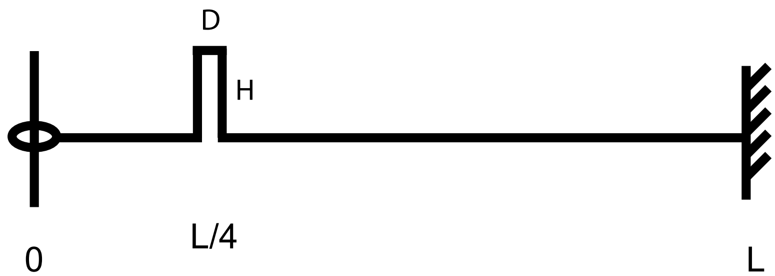

Figure 2: Deformed String

Consider a string of length L that is attached to a massless ring at $x = 0$ and to a fixed point at $x = L$. The ring is free to move perpendicular to the string along a frictionless rod. The tension of the string is T and the mass density is μ. The force of gravity can be ignored compared to other forces. At $t = 0$ the string is deformed such that there is a narrow rectangular pulse of height H as shown in the Figure 2. The width of the pulse D is much smaller than the length of the string L but it is finite. The center of the pulse is at $x_0 = L/4$. The string is initially stationary with $\frac{\partial y}{\partial t}(x, 0) = 0$.

a. Find the functional form, wave numbers k_m, frequencies ω_m and periods of oscillations τ_m of all possible normal modes of this string as a function of L, T, μ and integer m.

b. Find the expression for coefficients A_m in the expansion of $y(x, 0)$ into normal modes. Evaluate all integrals and simplify results. Express results as a function of L, H, D and m. You may find useful trigonometric expressions in the formula sheet.

c. Find which of the coefficients are equal to zero, if any.

d. Write the complete expression for the time-dependent Fourier decomposition of the pulse $y(x, t)$ in form of the infinite series of coefficients multiplied by the appropriate time-dependent terms.

e. Find the shape of the string at $t = \tau_1/2$ where τ_1 is the period of oscillations of the lowest normal mode. You do not have to use Fourier expansion, any logical argument would be acceptable.

Problem 4

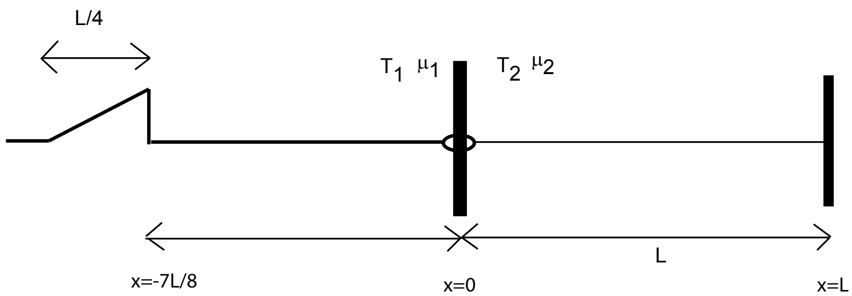

Figure 3: Traveling triangle

Consider two strings with mass density $\mu_1 = \mu$ and $\mu_2 = \mu/2$ and tensions $T_1 = T$ and $T_2 = T/2$ connected by a massless ring at $x = 0$. The ring can move along a frictionless rod. String 1 extends to the left, far beyond $x = -L$, and string 2 is attached to a fixed point on the wall at $x = L$. Initially a triangular pulse of width $L/4$ is moving along string 1 from left to right. At t=0 the front edge of the triangle is at $x = -7L/8$ as shown in Figure 3.

a. Write wave equations on both sides of the ring and specify the boundary conditions at $x = 0$.

b. Do you expect reflected waves at $x = 0$ and at $x = L$? Will the reflected wave have the same or the opposite sign than the incoming wave at $x = 0$ and at $x = L$?

c. Assume that the incoming pulse is of the form $f_1(x,t) = f_1(x/v_1 - t)$. Find the functional form of the transmitted $g(x,t)$ and reflected $f_2(x,t)$ pulses at $x = 0$ and $x = L$. Consider only times $t \leq 2L\sqrt{\frac{\mu}{T}}$.

d. Make a sketch of string deformations at $t = L\sqrt{\frac{\mu}{T}}$ and $t = 2L\sqrt{\frac{\mu}{T}}$.

Problem 5

A string of length L, mass M and tension T is fixed at both ends.

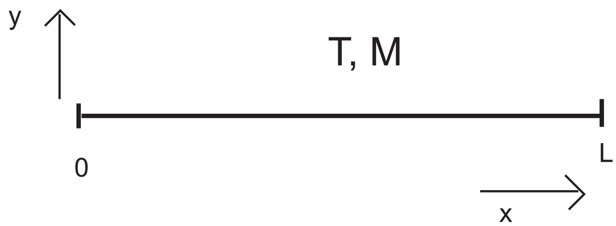

Figure 4: String attached at both ends

a). What is the wavelength λ_1, in the lowest possible normal mode, and what is the associate frequency ω_1?

b). What is the wavelength λ_n of the n^{th} normal mode, and what is the associated frequency ω_n?

We deform the string as shown in Figure 5 (the triangle is equilateral). The vertical displacement is highly exaggerated in the figure. In the interval $0 < x < L$ the shape of the string is given by $y = f(x)$. As per Fourier:

$$y(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \qquad (1)$$

with

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx \qquad (2)$$

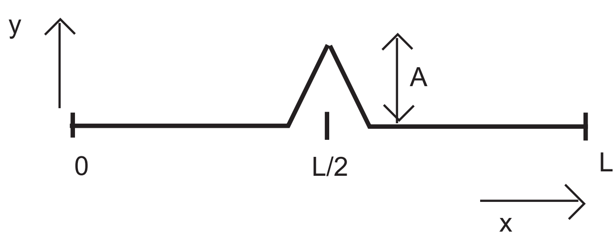

Figure 5: Triangle on the string

c). Which values for B_n will be zero ? Give your reasons. DO NOT attempt any integrations!

We release the plucked string with zero speed.

d). What is the minimum amount of time we have to wait for the string to look like Figure 4 ($y(x) = 0$), and what is the minimum time for the string to look again like Figure 5. Give your reasons.

e). Make a sketch of the string $\frac{1}{4}\sqrt{\frac{ML}{T}}$ seconds after release.

f). Let $t = 0$ be the moment that the string for the first time (after release) looks like Figure 4 ($y(x) = 0$). Let the B value of the 5^{th} normal mode be B_5 ($n = 5$). Do not attempt to calculate B_5! What is the evolution in time of this 5^{th} mode? In other words what is $y_5(x,t)$? The index 5 indicates the 5^{th} mode. Express your answer in terms of T, M, L and B_5.

Assume now that someone (NOT you !) has calculated all values of B_n. The person proudly shows you, on her laptop, by adding up the 20 allowed lowest modes, that the match with the shape shown in Figure 5 is near perfect. For your curiosity, you want to see the shape of the string for values of x from $L \to 3L$. The rope does not exist there, but equation (1) does not know that.

g). Sketch $f(x)$ in the range $0 \to 3L$.

We now let the end of the string ($x = L$) move freely without friction. We keep the tension the same using a "massless" ring that slides without friction along a vertical bar.

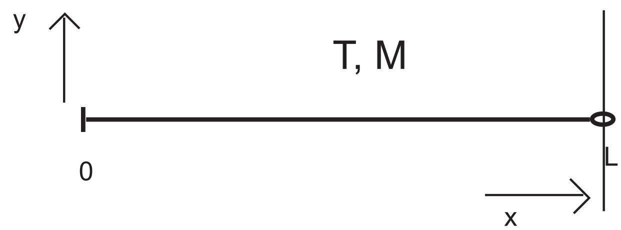

Figure 6: String with one free end

h). Answer question a) in this new configuration.

i). Answer question b) in this configuration.

We now deform the string as shown in Figure 5. Your friend, again proudly shows you the results of her Fourier analysis in the new configuration. Again you are curious about the range $L \to 3L$.

j). Sketch $f(x)$ in the range $0 \to 3L$.

Problem 6

Consider two perfectly conducting planes at $z = 0$ and $z = L$. Between the planes there is standing electromagnetic wave with E-field always along y-axis. The wave is described by an electric field $\vec{E}(z,t) = E_y(z,t)\hat{y}$, and a magnetic field $\vec{B}(z,t) = B_x(z,t)\hat{x}$.

a. Using boundary conditions for the electric field on the conducting planes, write down the expression for $E_y(z,t)$ in the n-th normal mode.

b. Using Maxwell's equations, or otherwise, find the corresponding magnetic field $B_x(z,t)$ in the n-th normal mode. *(Hint: you can consider a standing wave to be a superposition of two progressive waves)*

c. Derive expressions for the Poynting vector $\vec{S}$ and the densities U_E in the electric field and U_B in the magnetic field, as well as the time-averages of all three quantities.

d. For the second mode ($n = 2$), sketch the spatial variation of $\vec{E}$, $\vec{B}$, U_E and U_B between the two planes at $t = 0$.