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## Problem Set 9

*Due Thursday May 2 at 11.00AM*

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### Assigned Reading:

E&R	7 <sub>all</sub> , App <sub>M,N</sub>
Li.	8 <sub>2,3</sub> , 10 <sub>all</sub>
Ga.	4 <sub>c</sub> , 8 <sub>all</sub>
Sh.	13 <sub>all</sub>

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### 1. (10 points) Coulomb Potential Superposition States

Let  $\phi_{nlm}(\mathbf{r})$  denote the properly-normalized energy eigenfunctions of the coulomb potential with principle quantum number  $n$  and angular momentum quantum numbers  $l$  and  $m$ . Consider an electron in the state

$$\psi(\mathbf{r}) = C \left( \phi_{100}(\mathbf{r}) + 4i\phi_{210}(\mathbf{r}) - 2\sqrt{2}\phi_{21-1}(\mathbf{r}) \right)$$

- Find the normalization constant  $C$ .
- What is the expectation value of  $\hat{E}$ ?
- What is the expectation value of  $\hat{L}^2$ ?
- What is the expectation value of  $\hat{L}_z$ ?
- Write down  $\phi(\mathbf{r}, t)$  at some later time  $t$ .

*Hint: At no point should you use the functional form of these eigenstates! All you need is orthonormality of the energy eigenfunctions,  $(\phi_{nlm}|\phi_{n'l'm'}) = \delta_{nn'}\delta_{ll'}\delta_{mm'}$ , and the action of the various operators on the energy eigenstates.*

2. (20 points) **Where does a weakly bound electron live?**

Consider an electron trapped in a coulomb potential in the state corresponding to maximum orbital angular momentum,  $\ell = n - 1$ .

(a) Show that

$$\langle r \rangle = r_o n \left( n + \frac{1}{2} \right)$$

and that

$$\langle r^2 \rangle = r_o^2 n^2 (n + 1) \left( n + \frac{1}{2} \right)$$

(b) Use the above results to show that, for large values of  $n$  and  $\ell$ ,

$$\begin{aligned} \sqrt{\langle r^2 \rangle} &\rightarrow r_o n^2 \\ \frac{\Delta r}{\langle r \rangle} &\rightarrow 0 \\ E_n &\rightarrow -\frac{1}{2} \frac{e^2}{n^2 r_o} \end{aligned}$$

That is, show that, for large values of  $n$ , the electron is well localized near the surface of a sphere of radius  $n^2 r_o$  and has energy which is the same as that of a classical electron in a circular orbit of the same radius.

(c) What can you say about the size of a weakly bound state of hydrogen? Is this a quirk of hydrogen, or is this generally true of weakly bound states? Discuss?

3. (10 points) **Mighty Fine Structure**

Estimate the leading correction to the energy eigenvalues of Hydrogen due to relativistic corrections to the electron kinetic energy, as discussed in lecture. Use your results to make a diagram of the energies of the  $n=1, 2$  and  $3$  states both before and after including this “fine-structure” correction, indicating the numerical values the energies and remaining degeneracies of each energy eigenvalue.

**4. (20 points) Short Answer: Coulomb is not Hydrogen**

Answer each part with no more than one short sentence and no more than one equation. By “Hydrogen” here let’s agree to understand the simple model of an electron in the coulomb potential which we’ve studied in lecture so far.

- (a) Consider a “Hydrogen” atom in its  $(1,0,0)$  ground state. Is the electron in motion?  
*Hint: Classically, “In motion” means “where the particle is” varies with time. In Quantum Mechanics, the best answer we can give to the question, “where is the particle” is  $\langle x \rangle$ . So: in an energy eigenstate of Hydrogen, is the electron “in motion”?*
- (b) Why doesn’t a “Hydrogen” atom in its ground state radiate?
- (c) Consider a “Hydrogen” atom in its  $(2,1,1)$  excited state. Is the electron in motion?
- (d) Why doesn’t a “Hydrogen” atom in this excited state radiate?
- (e) Experimentally, if we excite a physical Hydrogen atom from  $(1,0,0)$  into  $(2,1,1)$ , it will relatively quickly decay back down to the ground state. How much angular momentum would our atom of “Hydrogen” emit in the process?
- (f) What wavelength will the resulting radiation have? How does this compare to the Rydberg-Ritz formula?
- (g) Experimentally, excited atoms radiate and decay to lower energy states. However, in (a)-(d) above, you (correctly) argued that the 8.04 model of “Hydrogen” does not radiate. There’s apparently a contradiction between your (correct) argument and the experimental fact that the atom decays to a lower energy state! What is missing in our model? In a more realistic model in which such a decay is possible, can the Energy operator possibly be Hermitian? Discuss.

### 5. (40 points) Lab Exercise II: Multiple Wells

Run the PhET simulation called “[Band Structure](#)” and use it to answer the following questions. Note that the numbers produced by this simulation are correct for a particle of mass  $m_e = .511MeV$ , the electron mass. *You are strongly encouraged to work in groups.*

- (a) Set the potential to 1 Square Well with Height  $18.2eV$  and Width  $0.4nm$ . Verify that there are 3 bound energy eigenstates, and that their energy eigenvalues are discrete and non-degenerate. Record these values and sketch each eigenstate.
- (b) Keeping the shape of the potential fixed, change the number of wells to  $N=2$ . Begin with the separation between wells set to the maximum allowed,  $0.2nm$ .
  - i. Verify that the energy eigenvalues are discrete, non-degenerate, and come in sets, or “bands”. How many states lie in each band? How does the mean energy of each band compare to the eigenenergies of the single well?
  - ii. For the top state and the bottom state in each band, count how many zeroes lie inside wells and how many lie between wells. How are these states related to the wave functions of the single well? What about the other states in the same band – how are they related to the wave functions of the single well?
  - iii. How does the width (in energy) of each band vary from band to band?
  - iv. Reduce the separation between wells, holding the width and height fixed. How do the width and mean energy of each band vary as you vary the separation?
  - v. Set to minimum separation. Within each band, one state looks nearly periodic with a single dominant period. Which state, and what is its (approximate) wavelength,  $\lambda$ ? Compute the energy of a free particle of mass  $m_e$  and this wavelength; compare this to the observed energy of the nearly-periodic state.
- (c) Keeping the shape of the potential fixed, repeat the above analysis for  $N=3, 4$ , and so on (*on scratch paper, no need to turn this in*) until you are ready to make predictions for the special case of  $N=10$ . State your predictions.
- (d) Keeping the shape of the potential fixed, change the number of wells to 10. Repeat each step of the analysis above. How did your predictions pan out?
- (e) For the case of 10 wells, make a plot of  $E_n$  as a function of  $n$  for the minimum allowed separation. Indicate on your plot the free-particle energies you calculated in part (b.v). Point out any interesting features of this graph. What would your graph look like if we set the separation to infinity? Sketch a prediction. What about sending the separation to zero? Again, sketch your prediction.
- (f) Keeping the shape of the potential fixed and setting the separation to the minimum allowed, imagine taking the number of wells very large,  $N \gg 1$ . What would your plot of  $E_n$  as a function of  $n$  look like? Sketch your prediction, compressing the horizontal axis so that all bound states appear in the plot.

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