

Problem 1. (10 points) Coulomb Potential Superposition States

(a) **(2 points)** Our wavefunction is given by

$$\psi = C \left(\phi_{100} + 4i\phi_{210} - 2\sqrt{2}\phi_{22-1} \right) \quad (1)$$

To normalize, we compute $\langle \psi | \psi \rangle$ and set it to 1:

$$\langle \psi | \psi \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} C^* \left(\phi_{100} + 4i\phi_{210} - 2\sqrt{2}\phi_{22-1} \right)^* \times \quad (2a)$$

$$C \left(\phi_{100} + 4i\phi_{210} - 2\sqrt{2}\phi_{22-1} \right) d^3\vec{r} \quad (2b)$$

$$= |C|^2 \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_{100}^* \phi_{100} d^3\vec{r} + \quad (2c)$$

$$+ 16 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_{210}^* \phi_{210} d^3\vec{r} + 8 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_{22-1}^* \phi_{22-1} d^3\vec{r} \right) \quad (2d)$$

$$= |C|^2 [(\phi_{100}|\phi_{100}) + 16(\phi_{210}|\phi_{210}) + 8(\phi_{22-1}|\phi_{22-1})] \quad (2e)$$

$$= |C|^2(1 + 16 + 8) = 25|C|^2, \quad (2f)$$

where we have used the fact that the energy eigenfunctions are normalized. Note that “cross-terms” do not appear because the energy eigenstates are orthogonal. Choosing C to be real gives

$$C = \frac{1}{5}. \quad (3)$$

(b) **(2 points)** Finding the expectation value of \hat{E} requires very similar manipulations, except this time we need to “sandwich” the energy operator in between the two copies of ψ :

$$\langle \hat{E} \rangle = \langle \psi | \hat{E} \psi \rangle. \quad (4)$$

Note that the cross-terms *still* do not appear, because the energy eigenstates satisfy the energy eigenvalue equation *i.e.*

$$\hat{E}\psi_{nlm} = E_n\psi_{nlm}, \quad (5)$$

so for example

$$\langle \psi_{100} | \hat{E} \psi_{311} \rangle = E_3 \langle \psi_{100} | \psi_{311} \rangle = 0. \quad (6)$$

Armed with this, we can immediately say

$$\hat{E} = (\psi|\hat{E}\psi) \quad (7a)$$

$$= |C|^2 [(\phi_{100}|\hat{E}\phi_{100}) + 16(\phi_{210}|\hat{E}\phi_{210}) + 8(\phi_{22-1}|\hat{E}\phi_{22-1})] \quad (7b)$$

$$= |C|^2 [E_1(\phi_{100}|\phi_{100}) + 16E_2(\phi_{210}|\phi_{210}) + 8E_2(\phi_{22-1}|\phi_{22-1})] \quad (7c)$$

$$= \frac{1}{51} (E_1 + 24E_2). \quad (7d)$$

(c) **(2 points)** The algebra required to find \hat{L}^2 is very similar to what we have already done:

$$\hat{L}^2 = (\psi|\hat{L}^2\psi) \quad (8a)$$

$$= |C|^2 [(\phi_{100}|\hat{L}^2\phi_{100}) + 16(\phi_{210}|\hat{L}^2\phi_{210}) + 8(\phi_{22-1}|\hat{L}^2\phi_{22-1})] \quad (8b)$$

$$= |C|^2 [(0\hbar^2)(\phi_{100}|\phi_{100}) + 16(2\hbar^2)(\phi_{210}|\phi_{210}) + 8(6\hbar^2)(\phi_{22-1}|\phi_{22-1})] \quad (8c)$$

$$= \frac{80\hbar^2}{25}. \quad (8d)$$

(d) **(2 points)** Finally, for \hat{L}_z we have

$$\hat{L}_z = (\psi|\hat{L}_z\psi) \quad (9a)$$

$$= |C|^2 [(\phi_{100}|\hat{L}_z\phi_{100}) + 16(\phi_{210}|\hat{L}_z\phi_{210}) + 8(\phi_{22-1}|\hat{L}_z\phi_{22-1})] \quad (9b)$$

$$= |C|^2 [(0\hbar)(\phi_{100}|\phi_{100}) + 16(0\hbar)(\phi_{210}|\phi_{210}) + 8(-\hbar)(\phi_{22-1}|\phi_{22-1})] \quad (9c)$$

$$= -\frac{8\hbar}{25}. \quad (9d)$$

(e) **(2 points)** Each energy eigenstate evolves in time by a phase:

$$\psi_{nlm}(\vec{r}, t) = \psi_{nlm}(\vec{r}, 0)e^{-iE_n t/\hbar}. \quad (10)$$

Note that the energy (and therefore the time evolution) depends only on the principal quantum number n . Using the superposition principle, we have

$$\psi(\vec{r}, t) = C(\phi_{100}(\vec{r})e^{-iE_1 t/\hbar} + 4i\phi_{210}(\vec{r})e^{-iE_2 t/\hbar} - 2\sqrt{2}\phi_{22-1}(\vec{r})e^{-iE_2 t/\hbar}), \quad (11)$$

where $E_n = -13.6 \text{ eV}/n^2$ since we are dealing with hydrogen.

Problem 2. (20 points) Where does a weakly bound electron live?

(a) **(10 points)** The general hydrogen eigenfunctions are given by:

$$\psi_{nlm}(r, \theta, \phi) = \left[\left(\frac{2}{nr_0} \right)^3 \frac{(n-l-1)!}{2n(n+l)!} \right]^{1/2} e^{-\frac{r}{nr_0}} \left(\frac{2r}{nr_0} \right)^l L_{n-l-1}^{2l+1} \left(\frac{2r}{nr_0} \right) Y_{lm}(\theta, \phi), \quad (12)$$

where L_p^k are the associated Laguerre polynomials. For the case when $l = n - 1$ the Laguerre polynomial in the above expression (12) becomes L_0^{2n-1} , but in general $L_0^k = 1$, so in this case the hydrogen eigenfunctions simplify to:

$$\psi_{n,n-1,m}(r, \theta, \phi) = \left[\left(\frac{2}{nr_0} \right)^3 \frac{1}{2n(2n-1)!} \right]^{1/2} e^{-\frac{r}{nr_0}} \left(\frac{2r}{nr_0} \right)^{n-1} Y_{n-1,m}(\theta, \phi). \quad (13)$$

From eq. (13) we determine that:

$$\begin{aligned} r &= \int \psi_{n,n-1,m}(r, \theta, \phi)^* r \psi_{n,n-1,m}(r, \theta, \phi) d^3r \\ &= \left[\left(\frac{2}{nr_0} \right)^3 \frac{1}{2n(2n-1)!} \right] \underbrace{\int_0^\infty r^3 \left(\frac{2r}{nr_0} \right)^{2n-2} e^{-\frac{2r}{nr_0}} dr}_{\left(\frac{2}{nr_0} \right)^{-4} (2n+1)!} \underbrace{\int Y_{n-1,m}(\theta, \phi)^* Y_{n-1,m}(\theta, \phi) d\Omega}_1 \\ &= nr_0 \left(n + \frac{1}{2} \right), \end{aligned} \quad (14)$$

and

$$\begin{aligned} \langle r^2 \rangle &= \int \psi_{n,n-1,m}(r, \theta, \phi)^* r^2 \psi_{n,n-1,m}(r, \theta, \phi) d^3r \\ &= \left[\left(\frac{2}{nr_0} \right)^3 \frac{1}{2n(2n-1)!} \right] \underbrace{\int_0^\infty r^4 \left(\frac{2r}{nr_0} \right)^{2n-2} e^{-\frac{2r}{nr_0}} dr}_{\left(\frac{2}{nr_0} \right)^{-5} (2n+2)!} \underbrace{\int Y_{n-1,m}(\theta, \phi)^* Y_{n-1,m}(\theta, \phi) d\Omega}_1 \\ &= (nr_0)^2 \left(n + \frac{1}{2} \right) (n + 1). \end{aligned} \quad (15)$$

(b) **(6 points)** From eq. (15) we get:

$$\sqrt{\langle r^2 \rangle} = n^2 r_0 \sqrt{1 + \frac{3}{2n} + \frac{1}{2n^2}} \xrightarrow{n \rightarrow \infty} n^2 r_0. \quad (16)$$

By definition the standard deviation (uncertainty) in r is:

$$\Delta r = \sqrt{\langle r^2 \rangle - \langle r \rangle^2} = \sqrt{(nr_0)^2 \left(n + \frac{1}{2}\right)(n+1) - (nr_0)^2 \left(n + \frac{1}{2}\right)^2} = \frac{nr_0}{\sqrt{2}}, \quad (17)$$

so the relative uncertainty in position becomes:

$$\frac{\Delta r}{\langle r \rangle} = \frac{nr_0}{\sqrt{2}nr_0 \left(n + \frac{1}{2}\right)} = \frac{1}{\sqrt{2} \left(n + \frac{1}{2}\right)} \xrightarrow{n \rightarrow \infty} 0. \quad (18)$$

In the expression for energy there are no powers of n which we can neglect for high n , so the energy retains its exact expression. The point of this exercise is to demonstrate the correspondence principle: for high quantum numbers the quantum systems start to behave classically. This kind of highly excited atoms are called Rydberg atoms.

- (c) **(4 points)** As shown in the previous part (b) the radius of the bound state increases as $n^2 r_0$. This behavior is specific to the long range potentials (actually infinite in the case of Coulomb potential). For example in the case of a potential well (finite range) the bound state is limited more or less to the width of the well.

Problem 3. (10 points) Mighty fine structure

Recall the relativistic correction to the kinetic energy that we have seen in class:

$$KE = \sqrt{m^2c^4 + \vec{p}^2c^2} - mc^2 \approx \frac{\vec{p}^2}{2m} - \frac{\vec{p}^4}{8m^3c^2},$$

where we assume that $\frac{|\vec{p}|}{mc} \ll 1$. The total energy operator then becomes

$$\hat{E} = \frac{\hat{p}^2}{2m} - \frac{\hat{p}^4}{8m^3c^2} - \frac{e^2}{r}. \quad (19)$$

We can estimate the correction to the energy by evaluating¹

$$\delta E_{nlm} = \langle \phi_{nlm} | \left(-\frac{\hat{p}^4}{8m^3c^2} \right) | \phi_{nlm} \rangle, \quad (21)$$

where ϕ_{nlm} are the hydrogen eigenstates that we worked out in class. Note that

$$\langle \phi_{nlm} | \hat{p}^4 | \phi_{nlm} \rangle = \langle \phi_{nlm} | \hat{p}^2 | \hat{p}^2 | \phi_{nlm} \rangle = (2m)^2 \langle \phi_{nlm} | (E_{nlm}^{(0)} - \hat{V}) | (E_{nlm}^{(0)} - \hat{V}) | \phi_{nlm} \rangle =$$

¹We can write (19) as

$$\hat{E} = \hat{E}_0 + \delta\hat{E},$$

where \hat{E}_0 is the Hydrogen Hamiltonian, and its eigenstates as

$$E = E_0 + \delta E,$$

where E_0 are the Hydrogen energy eigenvalues that we worked out in class. Similarly, we can also write

$$\psi = \phi + \delta\phi,$$

where ψ is an exact energy eigenstate of \hat{E} . Note that we expect all the quantities with the δ to be small. The Schödinger equation for eigenstates then becomes

$$(\hat{E}_0 + \delta\hat{E})(\phi + \delta\phi) = (E_0 + \delta E)(\phi + \delta\phi).$$

Dotting with $\langle \phi |$ and expanding, we obtain

$$\langle \phi | \hat{E}_0 \phi \rangle + \langle \phi | \delta \hat{E}_0 \phi \rangle + \langle \phi | \hat{E}_0 \delta \phi \rangle + \langle \phi | \delta \hat{E}_0 \delta \phi \rangle = E_0 + \delta E_0 + E_0 \langle \phi | \delta \phi \rangle + \delta E_0 \langle \phi | \delta \phi \rangle. \quad (20)$$

Note that the last terms in both members are much smaller than the others, as they contain two δ 's, so we can drop them. Moreover, from

$$\hat{E}_0 \phi = E_0 \phi,$$

and from the fact that \hat{E}_0 is a Hermitian operator, we can write

$$\langle \phi | \hat{E}_0 \delta \phi \rangle = \langle \hat{E}_0^\dagger \phi | \delta \phi \rangle = \langle \hat{E}_0 \phi | \delta \phi \rangle = E_0 \langle \phi | \delta \phi \rangle,$$

and therefore (20) gives

$$\langle \phi | \delta \hat{E}_0 \phi \rangle = \delta E_0.$$

$$= (2m)^2 (\phi_{nlm} | (E_{nlm}^{(0)} - \hat{V})^2 \phi_{nlm}),$$

where we used

$$\left(\frac{\hat{p}^2}{2m} + V \right) \phi_{nlm} = E_{nlm}^{(0)} \phi_{nlm}.$$

Therefore

$$(\phi_{nlm} | \hat{p}^4 \phi_{nlm}) = (2m)^2 (\phi_{nlm} | (E_{nlm}^{(0)})^2 \phi_{nlm}) - 2(2m)^2 (\phi_{nlm} | (E_{nlm}^{(0)} \hat{V}) \phi_{nlm}) + (2m)^2 (\phi_{nlm} | \hat{V}^2 \phi_{nlm}). \quad (22)$$

The first term in the above equation is just

$$(\phi_{nlm} | (E_{nlm}^{(0)})^2 \phi_{nlm}) = (E_{nlm}^{(0)})^2 (\phi_{nlm} | \phi_{nlm}) = (E_{nlm}^{(0)})^2,$$

and the second term is

$$-2(\phi_{nlm} | (E_{nlm}^{(0)} \hat{V}) \phi_{nlm}) = 2E_{nlm}^{(0)} (\phi_{nlm} | \frac{e^2}{r} \phi_{nlm}).$$

To evaluate the above expression, notice that

$$\frac{e^2}{r} = -\frac{e}{2} \frac{d}{de} \left(\frac{\bar{p}^2}{2m} - \frac{e^2}{r} \right), \quad (23)$$

and recall from class that

$$(\phi_{nlm} | \left(\frac{\bar{p}^2}{2m} - \frac{e^2}{r} \right) \phi_{nlm}) = -\frac{me^4}{2\hbar^2 n^2}.$$

Using this together with (23), we obtain

$$(\phi_{nlm} | \frac{e^2}{r} \phi_{nlm}) = \frac{me^4}{\hbar^2 n^2}.$$

We obtain the third term in (22) with a similar trick, but we need to do a little more work. Recall from class that for eigenstates of \hat{L}^2 , we can write the energy of the hydrogen atom as

$$\frac{\hat{p}_r^2}{2m} + \frac{\hbar^2 l(l+1)}{2m r^2} - \frac{e^2}{r},$$

whose associated eigenvalues can be written in terms of l by using $j_{\max} = l + 1 - n$. In this notation, we saw that the eigenvalues read

$$-\frac{me^4}{2\hbar^2 (j_{\max} + l + 1)^2}. \quad (24)$$

Now observe that

$$\frac{1}{r^2} = \frac{2m}{\hbar^2} \frac{1}{2l+1} \frac{d}{dl} \left(\frac{\hbar^2 l(l+1)}{2m r^2} \right) = \frac{1}{2l+1} \frac{d}{dl} \left(\frac{\hat{p}_r^2}{2m} + \frac{\hbar^2 l(l+1)}{2m r^2} - \frac{e^2}{r} \right),$$

and the expression in the bracket is precisely the energy operator of the hydrogen atom, for which we know the expectation value on ϕ_{nlm} . Therefore, using (24)

$$(\phi_{nlm} | \frac{1}{r^2} \phi_{nlm}) = \frac{2m}{\hbar^2} \frac{1}{2l+1} \frac{d}{dl} \left(-\frac{me^4}{2\hbar^2(j_{\max} + l + 1)^2} \right) = \frac{m^2 e^4}{\hbar^4(l + \frac{1}{2})n^3}.$$

Summing up everything, we obtain

$$(\phi_{nlm} | \hat{p}^4 \phi_{nlm}) = (2m)^2 \left(\frac{m^2 e^8}{4\hbar^4 n^4} - 2 \frac{me^4}{2\hbar^2 n^2} \frac{me^4}{\hbar^2 n^2} + \frac{m^2 e^8}{\hbar^4(l + \frac{1}{2})n^3} \right) = -\frac{3m^4 e^8}{\hbar^4 n^4} + \frac{4m^4 e^8}{\hbar^4(l + \frac{1}{2})n^3},$$

and the total energy is

$$E_{nlm} = -\frac{me^4}{2\hbar^2 n^2} + \frac{3me^8}{8c^2 \hbar^4 n^4} - \frac{me^8}{2\hbar^4 c^2(l + \frac{1}{2})n^3} = E_n + \frac{E_n^2}{2mc^2} \left(3 - \frac{4n}{(l + \frac{1}{2})} \right).$$

We can write this in terms of the so-called *fine structure constant* α ,

$$E_{nlm} = E_n + \frac{1}{2} mc^2 \alpha^4 \left(\frac{3}{4n^4} - \frac{1}{n^3(l + \frac{1}{2})} \right),$$

where

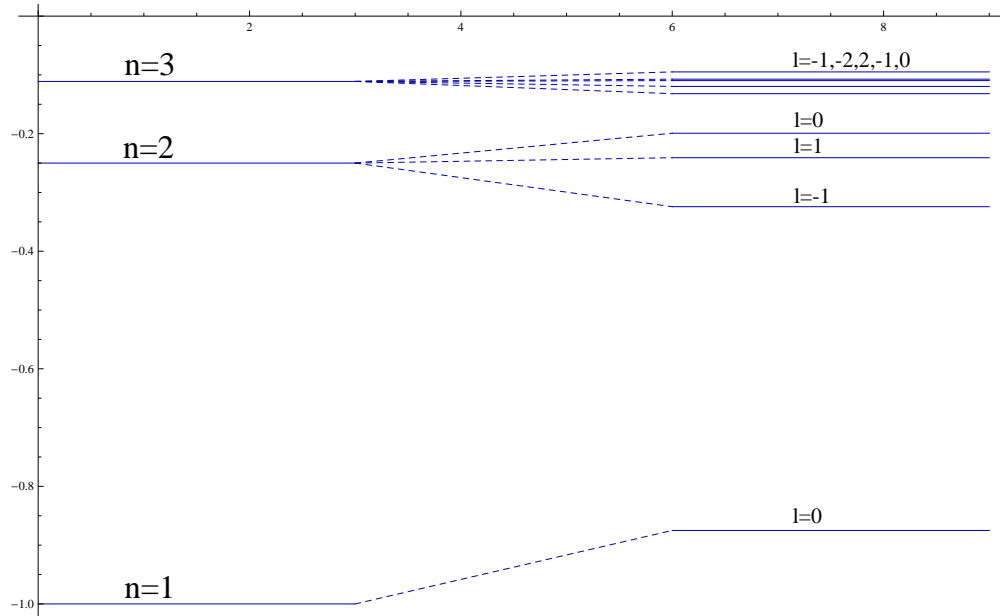
$$\alpha = \frac{e^2}{\hbar c}.$$

This dimensionless number is a fundamental constant of Physics, it appears every time we keep into account any quantum corrections in electrodynamics. In particular, it accounts for the corrections to the structures of the atomic spectra, and this is ultimately the reason why it is called “fine structure” constant.²

Note that to obtain this result we have used a series of trick. It is also possible to evaluate the integrals of $\frac{1}{r}$ and $\frac{1}{r^2}$ directly, although the calculation is very lengthy. A full treatment can be found on Bethe and Salpeter, “Quantum mechanics of one- and two-electron atoms”, 1977, p.17.

²For more about this, see R. Shankar, “Principles of Quantum Mechanics”.

Below is the diagram of the energy levels. On the left there are the unperturbed levels, on the right the corrected ones. The values of l for the $n = 3$ group goes from down to top (e.g. the lowest $n = 3$ level is $l = -1$).



The numerical values of the energy levels in eV are showed in the following table:

n	l	Before	After
1	0	-13.6057	-13.6066
2	-1	-3.40142	-3.40121
2	0		-3.40157
2	+1		-3.40145
3	-2	-1.51174	-1.51172
3	-1		-1.51168
3	0		-1.51179
3	+1		-1.51175
3	+2		-1.51175

As for each pair (n, l) the energy is different, the degeneracy is equal to the number of independent eigenstates with quantum numbers (n, l) . Since the only remaining quantum label is m , and $m = -l, \dots, l$, the degeneracy is $2l + 1$.

Problem 4. (20 points) Short Answers

- (a) **(3 points)** If the atom is in its $(n, l, m) = (1, 0, 0)$ state, its time evolution is given by

$$\Psi(\vec{r}, t) = \Psi_{100}(\vec{r})e^{-iE_1t/\hbar}. \quad (25)$$

Let's see what happens when we try to work out something like \hat{x} :

$$\hat{x} = \int_{-\infty}^{\infty} \Psi^* \hat{x} \Psi d^3\vec{r} = \int_{-\infty}^{\infty} \Psi_{100}^* e^{+iE_1t/\hbar} x \Psi_{100} e^{-iE_1t/\hbar} d^3\vec{r} = \int_{-\infty}^{\infty} \Psi_{100}^* x \Psi_{100} d^3\vec{r}. \quad (26)$$

The time-dependence cancels out! Given that dynamical quantities like \hat{x} do not vary in time, it would probably be best to say that the electron is *not* in motion³.

- (b) **(3 points)** Rough answer: as we saw in the previous part, the electron is not “in motion”, and thus not accelerating. Charges that are not in motion do not radiate.

More precise, but less satisfying answer: a hydrogen atom in its ground state does not radiate because if it did, the electron would have to drop to an even lower energy state. No such energy state exists. It is quantum mechanically forbidden, in the sense that it is impossible to construct a wavefunction consistent with a lower energy than the ground state.

- (c) **(2 points)** For precisely the same reason as in part (a), the electron is *not* “in motion”.
 (d) **(3 points)** Rough answer: same as rough answer in part (b).

More precise, but less satisfying answer: if our system is in an energy eigenstate, then it will remain in that eigenstate forever with its time evolution given by a multiplicative phase factor. It will not switch to a different eigenstate. This may mean that our model of “hydrogen” is too simplistic, since real atoms do not act in this way. The applicability of our model, however, is a separate debate. For our simple model, the Schrödinger equation tells us unambiguously that an atom in the $(2, 1, 1)$ excited state should remain there forever.

- (e) **(3 points)** The angular momentum of Hydrogen in the state (n, l, m) is given by

$$\hat{L}^2 \phi_{nlm} = \hbar^2 l(l+1) \phi_{nlm}.$$

By the conservation of angular momentum, if hydrogen drops from the $(2, 1, 1)$ state to the $(1, 0, 0)$ state, it will emit a photon of angular momentum

$$L_{l=1}^2 - L_{l=0}^2 = 2\hbar^2 = 87.12 \times 10^{-32} (eV \cdot s)^2. \quad (27)$$

³Even though from a rigorous standpoint it's not clear how one should define “in motion” from a quantum mechanical perspective. In quantum mechanics we simply have a wavefunction that extends over all space and evolves in time in a way that is specified by Schrödinger's equation.

(f) **(3 points)** The energy of a photon is given by

$$E = \frac{hc}{\lambda}. \quad (28)$$

Plugging the energy that we found in the previous part into this formula, we get

$$\lambda = 1216 \text{ \AA}, \quad (29)$$

which is a particularly important wavelength in astronomy! This agrees with the result from the Rydberg-Ritz formula.

(g) **(3 points)** Stationary states are, well, stationary. So they clearly can't radiate. We can of course build states in which the electron *does* move – superpositions! However, (a) they are necessarily periodic with period given by $1/$ the difference in energy of the terms in the superposition, and (b) the expectation value of the energy is necessarily time independent. So that also doesn't allow for radiation and decay.

If our atom really radiates, it must be true that the expectation value of the energy decays in time. However, we want the total probability that the electron is in *some* state to still be 1 – we want the energy to decrease, but we do not want the electron to disappear!! So we want the probability to look something like,

$$(E_{excited}) = e^{-\Gamma t}, \quad P(E_{ground}) = 1 - e^{-\Gamma t}.$$

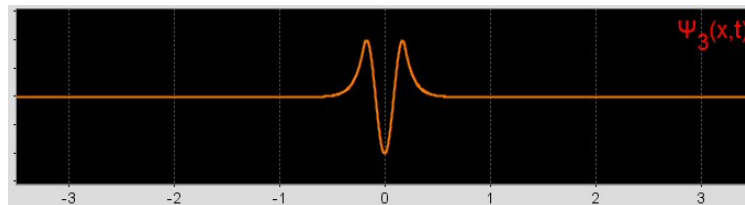
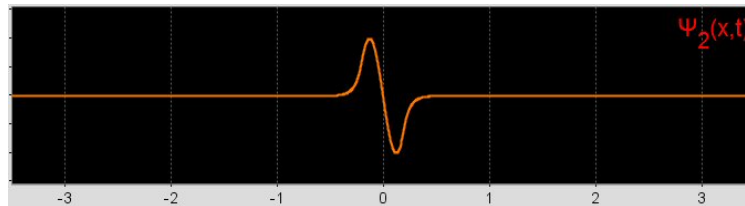
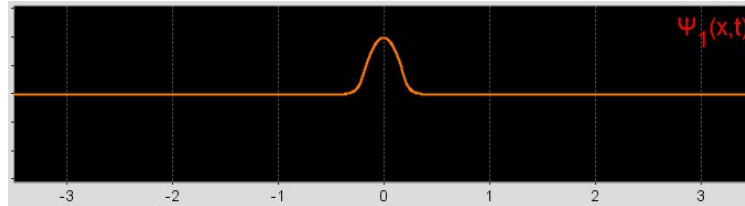
Meanwhile, we do not want the laws of physics to be time dependent, so the energy operator should be time-independent. Putting all that together, we want a time-independent energy operator which leads to decaying exponentials in the time evolution and a loss of the total energy. This is not compatible with unitary evolution by a hermitian energy operator.

So how do we explain the decay using quantum? Well, we have not introduced the coupling between the EM field and the atom. Properly, the quantum system we should study is the quantum atom interacting with the quantum EM field. This is obviously a harder problem. The lack of hermiticity we saw above is a consequence of treating the EM field classically and allowing it to absorb energy. Precisely how this works leads to quantum field theory and the theory of light-atom interactions; some very good discussions on this are available in the various quantum and light-atom books of Claude Cohen-Tannoudji.

Problem 5. (40 points) Lab Exercise: Multiple Wells

- (a) (2 points) $E_1 = 1.52eV$, $E_2 = 5.96eV$, $E_3 = 12.84eV$.

The corresponding three wavefunctions are shown below:



- (b) i. (2 points) We have 6 levels in total, they are non-degenerate and we can note from the graph that they are grouped in three sets. Each set, or band, has two energy levels, thus two states.

The mean energies of the three bands are: $\bar{E}_1 = 1.55eV$, $\bar{E}_2 = 6.085eV$, $\bar{E}_3 = 13.095eV$. These mean energies are higher than the energy levels that we found in part (a). Note that the differences $\bar{E}_n - E_n$ increase with n .

- ii. (2 points) The following table reports the number of nodes that lie within the wells + the number of nodes that lie within the barriers: States within the same band are roughly linear combinations of states from the single wells with the same energy level.

Note that the node theorem is respected, as the number of nodes is precisely equal to the excitation number.

- iii. (2 points) We have $\Delta E_1 = 0.02eV$, $\Delta E_2 = 0.11eV$, $\Delta E_3 = 0.45eV$, thus we observe that the energy width increases with the energy of the band.

level	1 st band	2 nd band	3 rd band
1	0+0	2+0	4+0
2	0+1	2+1	4+1

- iv. **(2 points)** If we decrease the separation between wells we see that the width of bands increases. This is expectable: decreasing the separation the wells feel more the presence of each other, so we expect the energy levels to look less similar to the single square well.
- v. **(2 points)** Given one band, the state which looks more periodic is the antisymmetric combination, i.e. the state with higher energy. Its approximate wavelength is roughly a divisor of twice the width of the well. Looking at the plot, for the three bands we have then $\lambda_1 = 0.9nm$, $\lambda_2 = 0.45nm$, $\lambda_3 = 0.225nm$. The corresponding energies of a free particle of mass m_e are given by

$$\mathcal{E}_i = \frac{h^2}{2m_e} \frac{1}{\lambda_i^2},$$

thus, $\mathcal{E}_1 = 1.86eV$, $\mathcal{E}_2 = 7.43eV$, $\mathcal{E}_3 = 16.72eV$. Note that these values are comparable to the highest excited levels of the three bands.

- (c) **(4 points)** With 10 wells, we expect to have again 3 bands, with 10 energy levels per band, and $\bar{E}_1 \gtrsim E_1$, $\bar{E}_2 \gtrsim E_2$, $\bar{E}_3 > E_3$. For the zeroes, we expect the obvious extension of the corresponding table we obtained before, for example in the first band we expect to have no nodes lying within the wells and a progressive number of nodes lying within the barriers from 0 to 9. The bands should be wider than the previous case. We also expect λ_i and \mathcal{E}_i to be the same as in the case of two wells.
- (d) **(8 points)** Our predictions of part (c) are confirmed, and we obtain the following values for the energies:

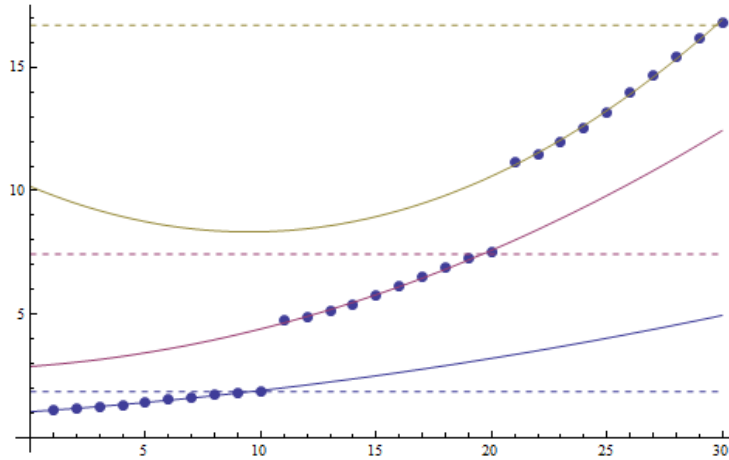
level	1 st band	2 nd band	3 rd band
1	1.14	4.73	11.17
2	1.17	4.87	11.48
3	1.24	5.14	11.99
4	1.31	5.41	12.54
5	1.4	5.73	13.2
6	1.52	6.15	13.96
7	1.63	6.53	14.69
8	1.72	6.88	15.42
9	1.82	7.25	16.21
10	1.87	7.49	16.79
\bar{E}	1.48	6.02	13.76
ΔE	0.73	2.76	5.62

and for the nodes:

level	1 st band	2 nd band	3 rd band
1	0+0	10+0	20+0
2	0+1	10+1	20+1
3	0+2	10+2	20+2
4	0+3	10+3	20+3
5	0+4	10+4	20+4
6	0+5	10+5	20+5
7	0+6	10+6	20+6
8	0+7	10+7	20+7
9	0+8	10+8	20+8
10	0+9	10+9	20+9

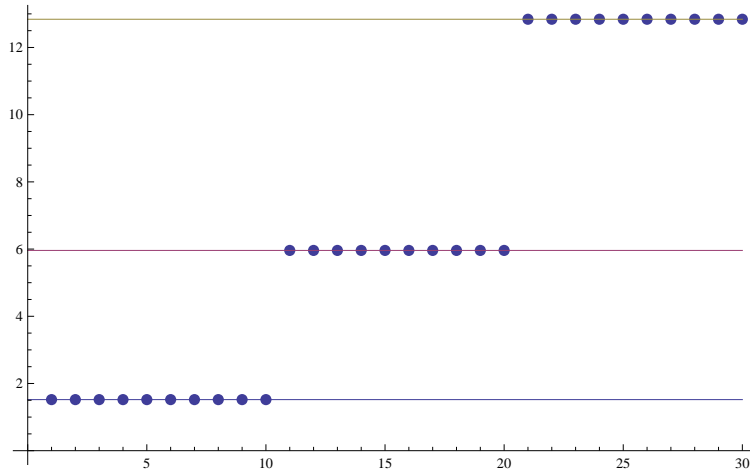
Moreover, λ_i and \mathcal{E}_i are the same as those obtained in part (b).

(e) **(8 points)** The plot is given in the following picture.

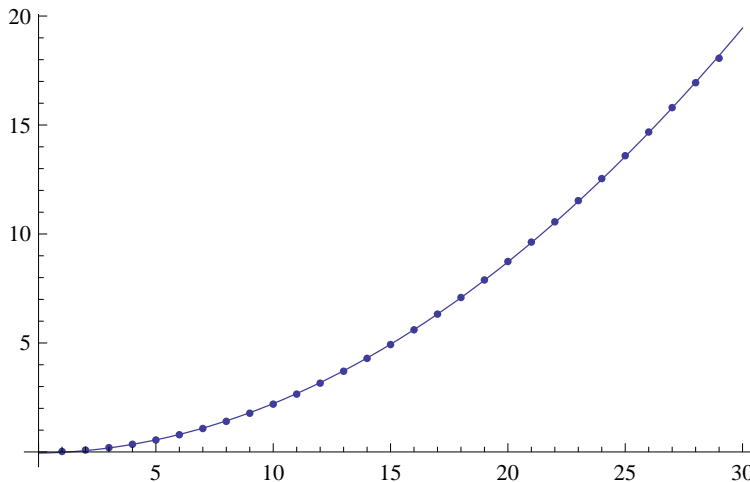


The dashed lines represent the free-particle energies, and the dots are the energy levels of the ten-well potential. Note that the energy levels group into three parabolic branches. Moreover, the free-particle energies perfectly coincide with the highest energy levels of each band.

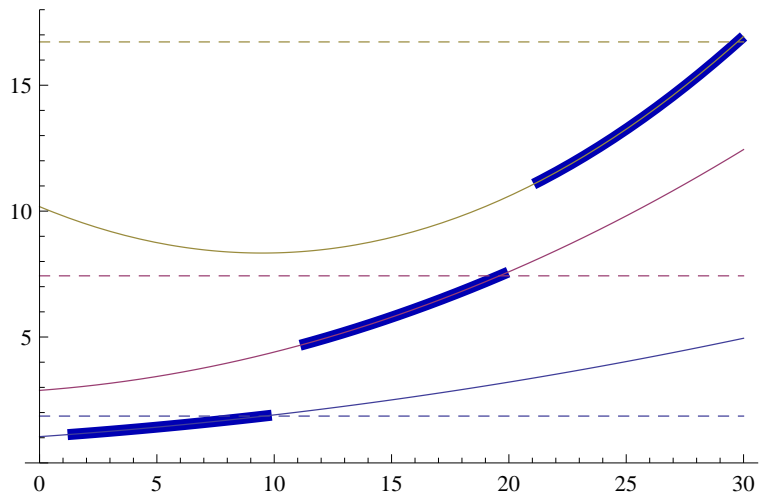
If we set the separation to infinity the wells do not feel the presence of each other, and the widths of the bands would shrink to zero, giving 10 degenerate energy levels per band, as is shown in the plot below.



If we send the separation to zero, the potential looks like the finite square well in part (a) with width 10 times longer. The energy eigenvalues will thus be those of a single large square well, without degeneracy. Note that they are distributed along a single parabolic branch. It turns out that the parabola it generates is very similar to the free-particle energy-momentum relation.



- (f) **(8 points)** As the number of wells N increases, the energy levels within a band tend to form a continuum, and in each band we see that $E = E_0 + \varepsilon n^2$, where E_0, ε are some constants, and $n = 1, \dots, N$. Note that since the energy dependence is quadratic in n , the electron can be seen as a free particle in a box (the larger N , the bigger the box), with an effective mass related to the aperture of the parabola that corresponds to the given band. Moreover, the free-particle energies perfectly coincide with the highest energy levels of each band. Below is a plot of the energy bands.



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Spring 2013

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