BARTON ZWIEBACH: Finite square well. So this brings us also to a little common aside. So far, we could find every solution. Now we're going to write the equations for the finite square well, and we're not going to be able to find the solution. But we're going to understand the solution. So you're going to enjoy a little-- mathematicians usually say it's the most important thing, understanding the solution. Finding it, it's no big deal.

But we're physicists as well. So we sometimes have to find the solutions. Even if we don't understand them very well, it's nice to find them. And then you're going to use numerical methods, and this is the part of the course where you're going to be using numerical methods a lot.

Here is the finite square well, and now we draw it symmetrically. Here it is. Here is x.

We're drawing the potential V of x. It extends from a to minus a. It's 0. This is the 0 of the potential. It's here.

And it goes down. The potential is negative here. Its value is minus V0 with V0 positive.

And then a possible energy for a bound state-- we're going to look for bound states. Bound states are normalizable states, normalizable solutions of the Schrodinger equation. So we're going to look for them. Look for bound states.

And they have energy less than 0. So something that has energy less than 0 is bound. You would have to give it some energy to put it at 0 energy, and the particle could escape.

The other thing about the bound state is that it will be some probability to find it here. Very little probability to find it in the forbidden region. And this energy, which is negative, it's somewhere here. We don't know what are the possible energies, but we will assume there's some energy that is negative, that is there. And that's the energy of the solution.

Let me write the equation again. The equation is psi double prime is minus 2m over h bar squared, E minus V of x times psi. This is the Schrodinger equation. You recognize it if you multiply h squared here, 2m there. And we've been solving already two examples where there was no potential, but finally, there is a potential.

So this is the energy of the particle. The energy of the particle can be interpreted as the potential energy plus the kinetic energy. Think intuitively here. If you have some energy over

here-- this is the potential energy. Well, all this much is kinetic energy.

On the other hand, in this region, the energy is smaller than the potential energy. So it has negative kinetic energy, which is classically not understandable. And in quantum mechanics, it just will have some probability of being here, but that probability will go down and eventually go to 0 in such a way that the wave function is normalizable.

So this is a very mysterious thing that happens here, that the wave function will not be just over here, but it will leak. And it leaks because there's a finite discontinuity. If you take the barrier to be infinitely high, it would leak so little that eventually it would not leak.

The wave function would vanish there. And it will be the end of the story, and you're back to the infinite square well. So the infinite square well is a limit of this as V0 goes to minus infinity.

OK, so a few numbers we can put in this graph. E, and this-- what is the energy difference here? It's E minus minus V0 , which is V0 plus E. And many times because E is negative, we'll write it as V0 minus absolute value of E. A negative number is equal to minus its absolute value.

So how about this whole constant here? Well, my tongue slipped, and I said this constant, but it's V of x. So what do you mean a constant? Well, the potential is piecewise constant.

So actually, we are not in such difficult situation because in this region, the potential is a constant. In this region, the potential is a constant. So this is the constant here, and we wish to understand what it is and what are the sines of this constant. This constant, that we call alpha-- see-- is going to be-- let's see what it is, the different circumstances.

Suppose you are here. The energy is bigger than the potential. So energy minus the potential is positive, and the constant is negative. So alpha is negative for x less than a. If it's negative and it's a constant, you're going to have trigonometric solutions for x, absolute value less than a, so trig.

On the other region, alpha will be positive for x greater than a, and you will have real exponentials of-- I'll just write exponentials. So E to the minus 3x, E to the 5x, things like that. This is the difference between trigonometric and real exponential solutions depend on the sines.

So we're going to have to impose boundary conditions as well because we're going to solve

the equation here inside with one value of alpha and then outside with another value of alpha. And then we're going to match them. So that's how this will go.

Now in this process, somehow the energy will be fixed to some value, some allowed values. There's a counting we could do to understand that, and we'll probably do it next time. At this moment, we'll just proceed. But we imagine there must be a quantization because in the limit as V0 goes to infinity and the potential well becomes infinitely deep, you're back to an infinite square well that has quantized energies that we calculated. So that should be quantized, and it should be no problem

OK, ready to do some work? Let's-- Yes?

AUDIENCE: So when alpha is 0 and you get polynomial solutions, does that case not matter?

BARTON ZWIEBACH: When alpha is 0-- well, alpha is going to be never 0. You see, either E is less than the potential or is better than the potential. So you're not going to get alpha equals 0.

You could say, do we have some energy maybe here or here? Well, you could have those energies, in which case you will see what happens. It's not quite polynomial solutions.

There's one property about potentials that actually is in the homework due this week, which is that there cannot be solutions of the Schrodinger equation with less energy than the minimum of the potential. You have to have more energy than the minimum of the potential. So alpha equal to 0 is not going to show up in our analysis.

OK, so let's begin. Moreover, we have to use this result. The potential is symmetric, completely symmetric, so there are going to be even solutions and odd solutions. So let's consider the even solutions. Solutions. So those are solutions. Psi of minus x is equal to psi of x.

Now, you will see how I solve this thing now, and the lesson of all of this is the relevance of unit-free numbers. Unit-free numbers are going to be your best friends in solving these equations. So it will look like I'm not solving anything except making more and more definitions, and all of the sudden, the solution will be there.

And it's a power of notation as well. This problem can be very messy if you don't have the right notation. If you were solving it alone, if we would stop the lecture now and you would go home, you would probably find something very messy for quite a bit. And then maybe you clean it up

little by little, and eventually, something nice shows up.

So here's what we're going to do. We're solving for x in between a and minus a, even solutions. So what does the equation look like? d second psi d x squared is minus 2m over h squared E minus V of x-- in that region, the potential is minus V0. So this is minus 2m over h squared, V0 plus E, which I write as minus absolute value of E, psi. And I forgot a psi here.

OK. V0 is positive. Minus V0 is going down, and V0 minus the absolute value of E is positive. So this whole quantity over here is positive.

And here it comes, first definition. I will define k squared to be that quantity-- 2m over h squared V0 minus absolute value of E. And that's greater than 0 in this region, as we're trying to solve, and I'll call this equation one.

And the differential equation has become psi double prime is minus k psi. So the solutions are simple. It's trigonometrics, as we said, and the only solution-- k squared, I'm sorry. k squared-- and the only solution that is possible, because it's a symmetric thing, is cosine of kx. So psi of x is going to be cosine of kx, and that will hold from a to minus a. End of story, actually, for that part of the potential.

You could ask-- one second. You could say, you're going to normalize these things, aren't you? Well, the fact that I won't normalize them, it's just a lot of work, and it's a little messy. But that's no problem.

You see, you're finding energy eigenstates, and by definition, solving the differential equation is not going to give you a normalization. So this is a good solution of the differential equation, and let's leave it at that. This we'll call solution two.

How about the region x greater than a? Well, what does the differential equation look like? Well, it looks like psi double prime is equal to minus 2, m over h squared, E psi because the potential is 0. V0 outside. x equals a.

And here again, you want to look at the equation and know the sine. So maybe better to put the absolute value here. So this is 2m absolute value of E over h squared psi.

And one more definition-- kappa squared is going to be 2mE over h squared. That's equation number three. The equation has become psi double prime equals kappa squared psi, and the solutions for that are exponentials. Solutions are psi-- goes like E to the plus minus kappa x.

You see the solution we're constructing is symmetric. It's even. So let's worry just about one side. If one side works, the other side will work as well.

So I will just write that for psi of x is equal to the minus kappa x. That's a solution for x greater than a. If I just did that, I would be making a mistake.

And the reason is that yes, I don't care about the normalization of the wave function, but by not putting a number here, I'm selecting some particular normalization. And The wave function must be continuous and satisfy all these nice things. So yes, here I can maybe not put a constant, but here, already, I must put a constant.

It may be needed to match the boundary conditions. I cannot ignore it here. So I must put the constant a that I don't know, and that's going to be equation number four.

Now, look at your definitions. k squared for a trigonometric, kappa-- many people use kappa for things that go with exponentials. But look, k squared and kappa squared satisfy a very nice relation. If you add them up-- k squared plus kappa squared-- the energy part cancels, and you get 2mV0, which is positive, over h squared.

Well, that's not so bad, but we want to keep defining things. How can I make this really nice? If it didn't have units, it would be much nicer. This is full of units.

k times a length has no units, and kappa times a length has no unit. So multiply by a squared, and you have k squared a squared plus kappa squared a squared is equal to 2mV0 a squared over h squared.

Now, say, well yes, this looks nice. So let's make the new definitions. So don't lose track of what we are doing.

We need to find the energy. That's basically what we want. What are the possible energies? And we already included two constants-- k and kappa- and they have these properties here.

So let me define psi to be kappa a, and it will be defined to be positive. Kappa is defined to be positive, and k is defined to be positive. eta, you will define it to be ka. a It's unit-free. No units.

And from that equation, now we have eta squared plus psi squared is equal to the right-hand side, which I will call z0 squared. So z0 squared-- that's another definition-- is 2mV0 a squared over h squared. This is the list of your definitions.

OK. What did we do? We traded kappa and k that control the behavior of the wave function-- k inside the well, kappa outside the well-- we traded them for eta and psi, unit-free. And a new constant came up, z0. What is z0? Well, z0 is a very interesting constant. It's a number that you can construct out of the parameters of your potential.

It involves V0 and the width. If V0 is very large, z0 is large. If the width is very big, z0 is big. If the potential is shallow or very narrow, z0 is small.

But the most important thing about z0 is that it will give you the number of bound states. If z0 is very big, it's a very deep potential, we'll have lots of bound states. If z0 is very shallow, there will be one bound state, no more. You will see it today.

But z0 controls the number of bound states. And this is its role, and it will be very important-- these dimensionless quantities and number. If somebody says, I have a potential with z0 equals 5, you can tell immediately three bound states or some number of bound states. That's the nice thing about z0. And this is a very nice-looking equation, this equation of a circle in the 8x psi plane.