PROFESSOR: Levinson's theorem, in terms of derivations, that we do in this course, this is probably the most subtle derivation of the semester. It's not difficult, but it's kind of interesting and a little subtle. And it's curious, because it relates to things that seem to be fairly unrelated. But the key thing that one has to do is you have to use something. How all of the sudden are you going to relate phase shifts to bound states?

The one thing you have to imagine is that if you have a potential and you have states of a potential, if the potential changes, the states change. But here comes something very nice. They never appear or disappear. And this is something probably you haven't thought about this all that much. Because you had, for example, a square, finite square well.

If you made this more deep, you've got more bound states. If you made it less deep, the bound states disappear. What does it mean the bound states disappear? Nothing, really can disappear.

What really is happening, if you have the bounds, the square well. You have a couple of bound states, say. But then you have an infinity of scatterings states. And as you make this potential less shallow, the last bound state is approaching here. And at one state, it changes identity and becomes a scatterings state, but it never gets lost.

And how, when you make this deeper and deeper you get more state, is the scattering state suddenly borrowing, lending you a state that goes down? The states never get lost or disappear. And you will, say, how could you demonstrate that? That sounds like science fiction, because, well, there's infinitely many states here. How do you know it borrowed one?

Well, you can do it by putting it in a very large box. And then the states here are going to be finitely countable and discrete. And then you can track and see indeed how the states become bound. So you'd never lose or gain states. And that's a very, very powerful statement about quantum states in a system.

So this is what we're going to need to prove Levinson's theorem. So Levinson theorem theorem-- so it relates a number of bound states of the potential to the excursion of the phase shift. So let's state it completely. It relates the number N of bound states of the potential to the excursion, excursion of the phase from E equals 0 to E equals infinity.

So in other words, it says that N is 1 over pi delta of 0 minus delta of infinity, a number of bound states of your potential is predicted by the behavior of the phase shift of scattering. So how do we do this? This is what we want to prove.

So consider, again, our potential of range R and 0 here and here is x. And I want you to be able to count states, but discovering states are a continuous set of states. So in order to count states, we're going to put a wall here, as well, at some big distance L much bigger than A than R.

And we're going, therefore-- now the states are going to be quantized. They're never going be quite scatterings states. They're going to look like scatterings states. But they're precisely in the way that they vanished at this point.

Now you would say, OK, that's already a little dangerous to be. Oh, you've changed the problem a little, yes. But we're going to do the analysis and see if the result depends on L. If it doesn't depend on L and L is very large, we'll take the limit this L goes to infinity. And we claim, we have answer.

So we argue that L is a regulator, regulator to avoid a continuum, continuum of states to avoid that continuum of states. All right, so let's begin to count. To count this thing, we start with the case where there is no potential again. And why is that? Because we're going to try to compare the situation with no potential to the situation with potential.

So imagine let V identical is 0, no potential and consider positive energy states. These are the only states that exist. There are no bound states, because the potential is 0.

Well, the solutions were found before, we mentioned that these are what we call phi effects or sine of kx. But now we require that phi of L is equal to 0, because we do have the wall there. And therefore, si of kl must be 0 and kl must be n pi and n is 1, 2 to infinity.

You know we manage with the wall to discretize this state, because the whole world is now an infinite, a very big box, not infinite, but very big. So you've discrete the state. The separations are microelectron volts, but they are discrete. You can count them.

And then with this state over here, we think of counting them. And you say, well, I can count them with n. So if I imagine the k line from 0 to 50, the other states are over, all the values of k. And they could say, well, I even want to see how many states there are in a little element dk.

And for the that you would have that dk taken a differential here is dn times pi. So of the number of states that there are in dk, dn-- let me right here-- dn equals l over pi dk is the number of positive energy states in dk. In the range dk, in the range of momentum, dk that little interval, there are that many positive energy eigenstates. So far, so good.

So now let's consider the real case. Repeat for the case there is some potential. Well, you would say, well, I don't know how to count. I have to solve the problem of when the potential makes a difference. But no, you do know how to count. So repeat for V different from 0.

This time we have a solution for x greater than R. We know the solution. This is our universal solution with the phase. So there you have the si effect is e to i delta sine kx plus delta of k. That's a solution.

Yes, you have the solution always. You just don't know what delta is. But you don't know what delta-- you don't need to know what delta is to prove the theorem, You just need to know it's there.

So here it goes. And this time the wall will also do the same thing. We'll demand the si of x vanishes for x equal L. So this time we get that kx-- no kL plus delta of k is equal to some other number n prime times pi multiple of pi. This phase-- this total phase has to be a multiple of pi.

And what is n prime? I don't know what is n prime? It is some integers. I don't know whether it starts from 1, 2, 3 or from 100 or whatever.

The only thing we care is that, again, taking a little differential, you have dk times L plus d delta, dk, times dk is equal to the dn prime times pi. We take an infinitesimal version of this equation, which again tells me how many positive-- all these states are positive energy states. They have k.

So all these are positive energy states. So from this equation, we get that dn prime is equal to L over pi dk plus 1 over pi d delta dk times dk, which gives me if I know the dk, again, the number of states that you have in that range of k, You see it's like momentum is now quantized.

So for any little range of momentum, you can tell how many states there are. And here it is how many states there are, positive energy states, positive. This is the number of positive energy states in dk with V different from 0, here is with V equal 0.