8.044 Statistical Physics I   Spring Term 2013

Problem Set #8
Due in hand-in box by 12:40 PM, Wednesday, April 10

Problem 1: Dust Grains in Space

Astronomers have discovered that there exist in the interstellar medium clouds of "dust grains" whose chemical composition may include silicates (like sand) or carbon-containing compounds (like graphite or silicon carbide). Evidence indicates that many of these grains have a needle like shape.

If the rotational motion of a dust grain is to be described in terms of its principal axes (1,2, and 3) the appropriate coordinates (angles) and canonically conjugate angular momenta are θ_1, θ_2, θ_3 and L_1, L_2, L_3. In terms of these variables the classical Hamiltonian for a single grain is given by

$$\mathcal{H} = \frac{1}{2I_1}L_1^2 + \frac{1}{2I_2}L_2^2 + \frac{1}{2I_3}L_3^2$$

where I_i is the moment of inertia about the ith principal axis. Assume that a dust cloud is in thermal equilibrium at a temperature T.

a) Find an analytic expression (no unevaluated integrals) for the joint probability density function for the canonical variables, $p(\theta_1, \theta_2, \theta_3, L_1, L_2, L_3)$, for a single dust grain.

b) Assume that axis 3 is parallel to the long axis of the grain and that $I_3 << I_1 = I_2$. Will the angular momentum of a dust grain be more likely to be parallel or perpendicular to the long (3) axis? Find a quantitative result to support your contention.

c) Find the rotational contribution of the grains to the entropy of a dust cloud containing N dust grains.

d) Does this model for the rotational motion of the dust grains obey the third law of thermodynamics? Explain the reasoning behind your answer.

e) Should this model of rotational motion exhibit energy gap behavior? Why?

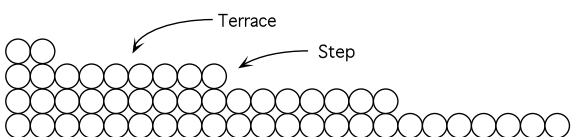**Problem 2:** Adsorption On a Stepped Surface

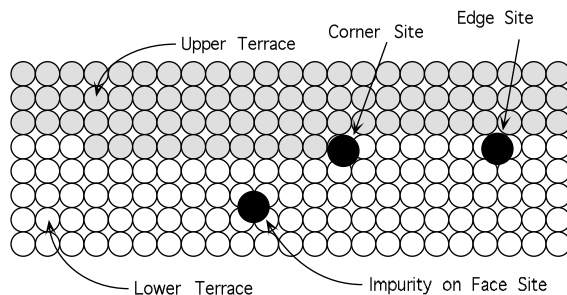Fig.1 Cut through crystal
perpendicular to the surface

Fig.2 Looking down on the
surface, showing adsorption sites

If a perfect crystal is cleaved along a symmetry direction, the resulting surface could expose a single geometrically flat plane of atoms. Alternatively, if the crystal is cut at a slight angle with respect to this direction, the resulting surface might take the form of a series of terraces of fixed width separated by steps of height corresponding to one atomic layer. This situation is illustrated in figure 1. The steps themselves may not be straight lines, but may have kinks when viewed from above as shown in figure 2. If impurity atoms were adsorbed on such a surface, their energy could depend on where they reside relative to the steps.

Consider N identical xenon atoms adsorbed on a silicon surface which has a total of M possible adsorption sites. There are three different types of site that the xenon could occupy. They would prefer to snuggle into a corner site at a kink in a step. One percent of the sites are corner sites, and their energy defines the zero of the energy scale for adsorbed atoms. Next in preference are edge sites. Fourteen percent of the M sites are edge sites with an energy Δ above that of a corner site. The majority (85%) of the adsorption sites are face sites, but they have an energy which is 1.5Δ above that of the corner sites. M is so large compared to N competition for a given site can be neglected; the xenon atoms can be considered completely independent. You may neglect the kinetic energy of the adsorbed atoms.

a) Find the partition function, $Z(N,T)$, for the xenon atoms in terms of the parameters M and Δ.

b) Find the ratio of xenon atoms on face sites to the number on corner sites.

c) Find an expression for the heat capacity of the adsorbed xenon atoms in the limit $kT \ll \Delta$.

d) What is the probability that a given xenon atom will be found on a face site in the limit $kT \gg \Delta$?

e) What is the limit of the entropy of this system as $T \to \infty$?

f) Should this model for adsorbed atoms exhibit energy gap behavior? Why?

Problem 3: Neutral Atom Trap

A gas of N indistinguishable classical non-interacting atoms is held in a neutral atom trap by a potential of the form $V(r) = ar$ where $r = (x^2 + y^2 + z^2)^{1/2}$. The gas is in thermal equilibrium at a temperature T.
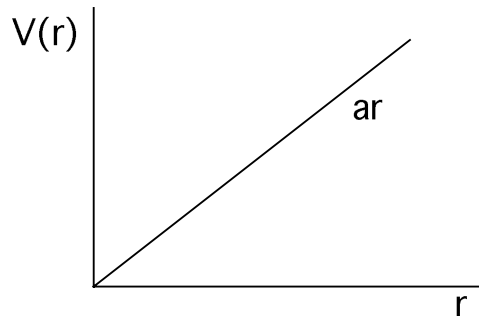
a) Find the single particle partition function Z_1 for a trapped atom. Express your answer in the form $Z_1 = AT^\alpha a^{-\eta}$. Find the prefactor A and the exponents α and η. [Hint: In spherical coordinates the volume element $dx\,dy\,dz$ is replaced by $r^2 \sin\theta\,dr\,d\theta\,d\phi$. A unit sphere subtends a solid angle of 4π steradians.]

b) Find the entropy of the gas in terms of N, k, and $Z_1(T, a)$. Do not leave any derivatives in your answer.

c) The gas can be cooled if the potential is lowered reversibly (by decreasing a) while no heat is allowed to be exchanged with the surroundings, $dQ = 0$. Under these conditions, find T as a function of a and the initial values T_0 and a_0.

Problem 4: Two-Dimensional H_2 Gas
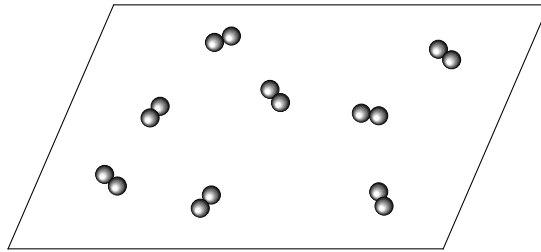
N molecules of molecular hydrogen H_2 adsorbed on a flat surface of area A are in thermal equilibrium at temperature T. On the surface they behave as a non-interacting two-dimensional gas. In particular, the rotational motion of a molecule is confined to the plane of the surface. The quantum state of the planar rotation is specified by a single quantum number m which can take on the values 0, ± 1, ± 2, ± 3, ± 4, etc. There is one quantum state for each allowed value of m. The energies of the rotational states are given by $\epsilon_m = (\hbar^2/2I)m^2$ where I is a moment of inertia.

a) Find an expression for the rotational partition function of a single molecule. Do not try to reduce it to an analytic function.

b) Find the ratio of the two probabilities $p(m = 3)/p(m = 2)$.

c) Find the probability that $m = 3$ given that $\epsilon = 9\hbar^2/2I$. Find the probability that $m = 1$ given that $\epsilon \le \hbar^2/2I$.

d) Find the rotational contribution to the internal energy of the gas in the high temperature limit where $kT \gg \hbar^2/2I$.

Problem 5: Why Stars Shine

The two major intellectual advances in physics at the beginning of the twentieth century were relativity and quantum mechanics. Ordinarily one associates relativity with high energies and great distances. The realm of quantum mechanics, on the other hand, is usually thought of as small distances or low temperatures. These perceptions are inaccurate. At the small end of the distance scale the energy levels of heavy atoms must be computed using relativity. For example, the electronic energy bands of lead show relativistic effects. At the large end of the distance scale QM can play an important role. We will show later in this course that the radius of a white dwarf or a neutron star depends on Planck's constant. The purpose of this homework problem is to show that QM is necessary even earlier in the life history of stars: they could not shine without it.

Relativity predated quantum mechanics. In the example discussed here, the physical source of the energy released by stars, this order of events gave rise to uncertainty (and some acrimony) in the scientific community.

In 1926 Arthur Eddington collected and reviewed all that was known about the interior of stars (*Internal Constitution of the Stars*, A. S. Eddington, Cambridge University Press, 1926). Relativity, in particular the equivalence of mass and energy, was well understood at that time. He concluded that the only possible source of the tremendous energy release in stars was the fusion of hydrogen nuclei into helium nuclei with the associated conversion of mass to energy.

Some physicists, however, questioned this view. They pointed out that whatever the chain of reactions leading from hydrogen to helium (we now recognize two, the proton-proton chain and the carbon cycle), the nuclei would have to surmount the repulsive coulomb barrier caused by their positive charges before any fusion step could take place. The temperature in the interior of the sun, 40 million degrees K, was not hot enough for this to happen.

a) Assume that the proton charge, $|e| = 4.8 \times 10^{-10}$ esu, is uniformly distributed throughout a sphere of radius $R = 1.2 \times 10^{-13}$ cm. What is the minimum energy E_{min} that another proton (taken to be a point particle) must have if it is to get within R of the first proton?

b) Find an expression for the probability p_+ that the kinetic energy of a particle in an ideal gas exceeds E_{min}. You may use the results of problem 4 in problem set 2. Assume that E_{min} is much greater than kT. Note that

$$\int_a^\infty \sqrt{y}e^{-y}\,dy \approx \sqrt{a}e^{-a} \quad \text{when } a \gg 1.$$

Evaluate p_+ for the value of E_{min} found above and a temperature of 40 million K.

c) The probability found above, 10^{-148}, does seem small; but, we should check its physical consequences just to be sure. In a gas the mean speed $<v>$, the mean free path L, and the mean free time τ are given by the following expressions.

$$<v> = \sqrt{8kT/\pi m} \quad L = 1/n\sigma \quad \tau = L/<v>$$

Here n is the number density of particles and σ is a collision cross-section. Assume that $\sigma = \pi(2R)^2$, that the sun is composed primarily of protons, and use a mass density of 100 g-cm^{-3} for the center of the sun. Find $<v>$, L, and τ at the center of the sun.

d) Now assume that the probability of fusion of two protons during a collision is independent of past history and given by p_+. What is the mean time to a fusion collision for a given proton? Compare this to the age of the Universe, 15 billion years. The mass of the sun is 2×10^{33} g. If all this mass were due to protons at the central density, how many fusion events would take place per second in the sun?

Eddington was aware of these arguments, but he was nevertheless convinced that the energy source must be fusion:

> "The difference of temperature between terrestrial and stellar conditions seems quite inadequate to account for any appreciable simulation of transmutation or annihilation of matter; and this is the chief ground on which censorship of our theories is likely. For example, it is held that the formation of helium from hydrogen would not be appreciably accelerated at stellar temperatures, and therefore must be ruled out as a source of stellar energy. But the helium which we handle must have been put together at some time and some place. We do not argue with the critic who urges that the stars are not hot enough for this process; we tell him to go and find *a hotter place*."

> A.S. Eddington (1926) - *The Internal Constitution of the Stars* (p. 301)

Quantum mechanics was developing quickly at this time, and it was being used to treat numerous physical problems. In 1928 Gamow showed that QM tunneling through the coulomb barrier could explain the radioactive decay of nuclei. In 1929 Atkinson and Houtermans (R. d'E. Atkinson and F. G. Houtermans, Zeitschrift für Physik, 54, 656 (1929)) showed that tunneling from the outside of the coulomb barrier could explain why fusion can take place at the calculated temperatures of stellar interiors.

Tunneling is now a familiar part of our understanding of QM. Simple problems involving rectangular barriers are done in 8.04. Tunneling through the coulomb barrier is a bit more involved mathematically; it is normally treated using the WKB approximation. We will not carry out such a calculation here. The resulting probability of getting through the barrier, when used in the computation done above, actually gives too high a fusion rate in the sun. A realistic discussion of the fusion rate would involve three other factors, each of which reduces the rate:

- Once a proton is in a light nucleus its probability of inducing fusion is still small compared to its probability of tunneling out again.

- The full fusion cycle involves a number of individual fusion events (5 for the pp chain, 4 for the carbon cycle).

- Not all of the protons in the sun are in the central region where the temperature and density are high enough to sustain the fusion process.

The concept of quantum tunneling had a major impact on both physics and astrophysics. Therefore it is surprising that Eddington made the following comment at the beginning of the 1930 edition of his book.

NOTE TO THE SECOND IMPRESSION

THE advances made since this book was first published are scarcely of sufficient importance to justify an extensive revision, and (except for correction of a few misprints) the text has been reprinted unchanged.

The actual fusion cycles themselves were worked out by Hans Bethe beginning in 1938. He received the Nobel Prize in 1967 for this work as well as other contributions to the theory of nuclear reactions.

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