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PROFESSOR:

Now, a theorem that was quite powerful and applied to complex vector spaces for old V longing to V complex vector space. This implied that the operator was zero, and it's not true for real vector spaces. And we gave a proof that made it clear that indeed, a proof wouldn't work for a real vector space.

I will ask you in the homework to do something that presumably would be the first thing you would do if you had to try to understand why this is true-- take two by two matrices, and just see why it has to be 0 in one case, and it doesn't have to be 0 in the other case. And I think that will give you a better perspective on why this happens.

And then once you do it for a two by two, and you see how it works, you can do it for n by n matrices, and it will be clear as well. So we'll use this theorem. Our immediate application of this theorem was a well-known result that we could prove now rigorously-- that t is equal to t dagger, which is to say the operator is equal to the adjointed.

And I think I will call this-- and in the notes you will see always the adjoint, as opposed to Hermitian conjugate. And I will say whenever the operator is equal to the adjoint, that it is Hermitian. So a Hermitian operator is equivalent to saying that $v^T T v$ is a real number for all v . And we proved that.

In other words, physically, Hermitian operators have real expectation values. This is an expectation value, because-- as you remember in the bracket notation-- $v^T T v$, you can write it $v^T T v$ -- the same thing.

So it is an expectation value, so it's an important thing, because we usually deal with Hermitian operators, and we want expectation values of Hermitian operators to be

real.

Now that we're talking about Hermitian operators, I delay a complete discussion of diagonalization, and diagonalization of several operators simultaneously, for the next lecture. Today, I want to move forward a bit and do some other things. And this way we spread out a little more the math, and we can begin to look more at physical issues, and how they apply here.

But at any rate, we're just here already, and we can prove two basic things that are the kinds of things that an 805 student should be able to prove at any time. They're really very simple. And it's a kind of proof that is very short-- couple of lines-- something you should be able to reproduce at any time.

So the first theorem says the eigenvalues of Hermitian operators-- H is for Hermitian-- are real. And I will do a little bit of notation, in which I will start with an expression, and evaluate it sort of to the left and to the right. So when you have an equation, you start here, and you start evaluating there.

So I will start with this-- consider $v^T T v$. And I will box it as being the origin, and I will start evaluating it. Now if v is an eigenvector-- so let v be an eigenvector so that $T v$ is equal to λv .

And now we say consider that expression in the box. And you try to evaluate it. So one way to evaluate it-- I evaluate it to the left, and then evaluate to the right. Is the naive evaluation-- $v^T T v$ is $\lambda v^T v$, so substitute it there. $v^T \lambda v$ -- we know it. And then by homogeneity, this λ goes out, and therefore it's $\lambda v^T v$.

On the other hand, we have that we can go to the right. And what is the way you move an operator to the first position? By putting a dagger. That's a definition. So this is by definition.

Now, we use that the operator is Hermitian. So this is equal to $(T v)^T v$. And this is by T Hermitian. Then you can apply again the equation of the eigenvalues, so this is $\lambda v^T v$. And by conjugate homogeneity of the first input, this is $\lambda^* v^T v$.

So at the end of the day, you have something on the extreme left, and something on the extreme right. v -- if there is an eigenvector, v can be assumed to be non-zero. The way we are saying things in a sense, 0 -- we also think of it as an eigenvector, but it's a trivial one.

But the fact that there's an eigenvalue means there's a non-zero v that solves this equation. So we're using that non-zero v . And therefore, this is a number that is non-zero.

You bring one to the other side, and you have $\lambda - \lambda^* \text{ times } v = 0$. This is different from 0 . This is different from 0 . And therefore, λ is equal to λ^* .

So it's a classic proof-- relatively straightforward. The second theorem is as simple to prove. And it's already interesting. And it states that different eigenvectors of a Hermitian operator-- well, different eigenvalues of Hermitian operators correspond to orthogonal eigenfunctions, or eigenvectors.

So different eigenvalues of Hermitian ops correspond to orthogonal eigenfunctions-- eigenvectors, I'm sorry. So what are we saying here? We're saying that suppose you have a v_1 that T gives you a $\lambda_1 v_1$. That's one eigenvalue.

You have another one-- v_2 is equal to $\lambda_2 v_2$, and λ_1 is different from λ_2 . Now, just focusing on a fact that is going to show up later, is going to make life interesting, is that some eigenvalues may have a multiplicity of eigenvectors. In other words, if a vector v is an eigenvector, $2v$ is an eigenvector, $\sqrt{3}v$ is an eigenvector, but that's a one-dimensional subspace.

But sometimes for a given eigenvalue, there may be a higher dimensional subspace of eigenvectors. That's a problem of degeneracy, and it's very interesting-- makes life really interesting in quantum mechanics. So if you have degeneracy, and that set of eigenvectors form a subspace, and you can choose a basis, and you could have several vectors here.

Now what do you do in that case? The theorem doesn't say much, so it means choose any one. If you had the bases there, choose any one. The fact remains that if these two eigenvalues are different, then you will be able to show that the eigenvectors are orthogonal. So if you have some space of eigenvectors-- a [INAUDIBLE] higher-dimensional space of eigenvectors, one eigenvalue, and another space with another-- any vector here is orthogonal to any vector there.

So how do you show this? How do you show this property? Well you have to involve v_1 and v_2 , so you're never going to be using the property that gives Hermitian, unless you have an inner product. So if you don't have any idea how to prove that, you presumably at some stage realize that you probably have to use an inner product. And we should mix the vectors, so maybe a v_2 inner product with this. So we'll take a v_2 inner product with $T v_1$.

And this is interesting, because we can use it, that $T v_1$ is $\lambda_1 v_1$ to show that this is just $\lambda_1 v_2 v_1$. And that already brings all kinds of good things. You're interested in this inner product. You want to show it's 0, so it shows up. So it's a good idea. So we have evaluated this, and now you have to think of evaluating it in a different way.

Again, the operator is Hermitian, so it's asking you to move it to the other side and exploit to that. So we'll move it to the other side a little quicker this time. It goes as T dagger, but T dagger is equal to T , because it's Hermitian. So this is the center of the equation. We go one way. We go the other way-- this time down.

So we'll put $T v_2 v_1$, and this is equal to $\lambda_2 v_2 v_1$. Your impulse should be it goes out as λ_2^* , but the eigenvalues are already real, so it goes out as $\lambda_2 v_2 v_1$, because the operator is Hermitian.

So at this moment, you have these two equations. You bring, say, this to the right-hand side, and you get $\lambda_1 v_1 v_2 - \lambda_2 v_1 v_2$ is equal to 0. And since the eigenvalues are supposed to be different, you conclude that v_1 inner product with v_2 is 0. So that's the end of the proof.

And those are the two properties that are very quickly proven with rather little effort. So where do we go from now? Well there's one more class of operators that are crucial in the physics. They are perhaps as important as the Hermitian operators, if not more.

They are some operators that are called unitary operators, and the way I will introduce them is as follows-- so I will say-- it's an economical way to introduce them-- so we'll talk about unitary operators.

If S is unitary, and mathematicians call it an isometry-- if you find that S acting on any vector-- if you take the norm, it's equal to the norm of the vector for all u in the vector space. So let's follow this, and make a couple of comments.

An example-- a trivial example-- this operator λ times the identity. λ times the identity acts on vectors. What does it do, λ times identity? The identity does nothing on the vector, and λ stretches it.

So λ , in order not to change the length of any vector should be kind of 1. Well, in fact, it suffices-- it's unitary-- if the absolute value of λ is equal to 1.

Because then λ is a phase, and it just rotates the vector. Or in other words, you know that the norm of λv is equal to the absolute value of λ times the norm of v , where this is a number.

And remember these two norms are different. This is the norm of a vector. This is the norm of a complex number. And therefore, if you take λu -- norm-- is λu is equal absolute value of λu , and absolute value of λ is equal to 1 is the answer. So that's a simple unitary operator, but an important one.

Another observation-- what are the vectors annihilated by this operator u ? Zero-- it's the only vector, because any other vector that's nonzero has some length, so it's not killed. So it kills only zero. So the null space of S is equal to the 0 vector.

So this operator has no kernel, nothing nontrivial is put to zero. It's an invertible

operator. So S is invertible. So that's a few things that you get very cheaply.

Now from this equation, Su equals u -- if you square that equation, you would have $SuSu$ is equal to uu . Maybe I should probably call it v . I don't know why I called it u , but let's stick to u .

Now, remember that we can move operators from one side to the other. So I'll move this one to that side. If you move an S here, you would put an S^\dagger . But since the dagger of an S^\dagger is S , you can move also the S to that side as S^\dagger . So uS^\dagger, Su -- you see that. If you want to move this one, you can move it by putting another dagger, and you get that one.

And this is uu , and therefore you get uS^\dagger, S minus the identity acting on u is equal to 0 for all u . So for every vector, this is true, because this is true. We just squared it.

And now you have our favorite theorem, that says if this is true in a complex vector space, this is 0, and therefore, you've shown that $S^\dagger S$ is equal to 1. So that's another property of unitary operators. In fact that's the way it's many times defined.

Unitary operators sometimes are said to be operators whose inverse is S^\dagger . I will not go into the subtleties of what steps in all these things I'm saying are true or not true for infinite dimensional operators-- infinite dimensional vector spaces. So I will assume, and it will be true in our examples, that if S^\dagger is an inverse from the left, it's also an inverse from the right. And perhaps everything is true for infinite dimensional vector spaces, but I'm not 100% positive.

So S^\dagger is the inverse of S . And that's a pretty important thing.

So one last comment on unitary operators has to do with basis. So suppose you have an orthonormal basis, e_1 up to e_n . Now you can define another basis. f_i equal-- I'll change to a letter U -- Ue_i where U is unitary, so it's like the S . In fact, most books in physics call it U for unitary. So maybe I should have changed that letter in there, too, as well.

So suppose you change basis. You put-- oh, there was something else I wanted to say before. Thanks to this equation, consider now the following thing-- $S U S^{-1} v$. $S U S^{-1} v$ -- you can move this S , for example, to the other side-- $S^{-1} S U v$, and $S^{-1} S$ is equal to 1, and it's $U v$.

So this is a pretty nice property. We started from the fact that it preserved the norm of a single vector, of all vectors, and now you see that in fact, it preserved the inner product. So if you have two vectors, to compare their inner product, compute them after action with U or before action with U , and it doesn't make a difference.

So suppose you define a second basis here. You have one orthonormal basis. You define basis vectors like this.

Then the claim is that the f_1 up to f_n is orthonormal. And for that you simply do the following-- you just check $f_i \cdot f_j$ is equal to $U e_i \cdot U e_j$. By this property, you can delete both U 's, rules, and therefore this is $e_i \cdot e_j$. And that's δ_{ij} . So the new basis is orthonormal.

If you play with these things, it's easy to get some extra curious fact here. Let's think of the matrix representation of the operator U . Well, we know how these things are, and let's think of this in the basis e basis.

So $U e_k$ is equal to $e_j U_{jk}$. That's the definition of U in the basis e -- the matrix elements of U . You can try to figure out what is U_{ki} in the f basis. How does operator U look in the f basis?

Well, let's just do it without thinking. So in the f basis, I would put $f_k \cdot U f_i$. Well, but f_k is $U e_k$, so I'll put $U e_k \cdot U f_i$. Now we can delete both U 's, and it's $e_k \cdot f_i$. And I can remember what f_i was, which is $e_j U_{ji}$.

And it's just the same as the one we had there. So the operator, unitary operator, looks the same in both bases. That might seem strange or a coincidence, but it's not. So I leave it to you to think about, and visualize why did that happen. What's the reason?

So the bracket notation-- we've been using it here and there-- and I will ask you to please read the notes. The notes will be posted this afternoon, and they will have-- not maybe all we've done today, but they will have some of what we'll do today, and all of what we've been doing. And the way it's done-- it's first done in this sort of inner product language, and then things are done in the bracket language.

And it's a little repetitious, and I'm trying to take out some things here and there, so it's less repetitious. But at this moment it's probably worth reading it, and reading it again. Yes.

AUDIENCE: [INAUDIBLE] if you have two orthonormal bases, is the transformation between them necessarily unitary?

PROFESSOR: Yes, yes. All right. So as I was saying we're going to go into the Dirac notation again. And here's an example of a place where everybody, I think, tends to use Dirac notation. And the reason is a little curious, and you will appreciate it quite fast.

So this will be the case of where we return to x and p operators, on a non-denumerable basis. So we're going to try to do x and p . now this is the classic of Dirac notation. It's probably-- as I said-- the place where everybody likes to use Dirac notation.

And the reason it's efficient is because it prevents you from confusing two things. So I've written in the notes, and we have all these v 's that belong to the vector space. And then we put this, and we still say it belongs to the vector space. And this is just a decoration that doesn't do much.

And we can play with this. Now, in the non-denumerable basis, the catch-- and the possible confusion-- is that the label is not quite a vector in the vector space. So that is the reason why the notation is helpful, because it helps you distinguish two things that you could confuse.

So here we go. We're going to talk about coordinate x , and the x operator, and the states. Well, this is a state space. So what kind of states do we have here? Well, we've talked about wave functions, and we could give the value of the wave function

of different places.

We're going to go for a more intrinsic definition. We're going to try to introduce position states. And position states will be called this-- x . Now, what is the meaning of this position state? We should think of this intuitively as a particle at x .

Now here's how you can go wrong with this thing, if you stop thinking for a second. What is, then, ax ? Is it ax , a being a number. Is it the same thing?

No, not at all. This is a particle at the coordinate ax , and this is a particle at x with some different amplitude-- very different. So this is not true-- typical mistake. This is not minus x . That's totally different. So there's no such thing as this, either.

It doesn't mean anything. And the reason is that these things are not our vectors. Our vector is this whole thing that says a particle at x . Maybe to make a clearer impression, imagine you're in three dimensions, and you have an x vector. So then you have to ket this. This is the ket particle at x . x is now a vector. It's a three-dimensional vector.

This is a vector, but it's a vector in an infinite dimensional space, because the particle can be anywhere. So this is a vector in quantum mechanics. This is a complex vector space. This is a real vector space, and it's the label here.

So again, minus x is not minus x vector. It's not the vector. The addition of the bra has moved you from vectors that you're familiar with, to states that are a little more abstract. So the reason this notation is quite good is because this is the number, but this i --- or this is a coordinate, and this is a vector already.

So these are going to be our basis states, and they are non-denumerable. And here you can have that all x must belong to the real numbers, because we have particles in a line, while this thing can be changed by real numbers. The states can be multiplied by complex numbers, because we're doing quantum mechanics.

So if you want to define a vector space-- now, this is all infinite dimension. It's a little worse in this sense the basis is non-denumerable. If I use this basis, I cannot make

a list of all the basis vectors. So for an inner product, we will take the following-- we will take x with y to be $\delta(x - y)$. That will be our inner product. And it has all the properties of the inner product that we may want.

And what else? Well at this moment, we can try to-- this is physically sensible, let me say, because if you have a particle at one point and a particle at another point, the amplitude that this particle at one point is at this other point is 0. And these states are not normalizable. They correspond to a particle at the point, so once you try to normalize them, you get infinity, and you can't do much.

But what you can do here is state more of the properties, and learn how to manipulate this. So remember we had one was the sum of all $e_i e_i$. The unit operator was that.

Well, let's try to write a similar one. The unit operator will be the sum over all x 's. And you could say, well, looks reasonable, but maybe there's a $1/2$ in here, or some factor. Well, no factor is needed. You can check that-- that you've defined this thing properly. So let me do it.

So act to on this so-called resolution of the identity with the vector y , so 1 on y is equal to y . And now let's add on the right $x y$. This is $\delta(x - y)$. And then when you integrate, you get y . So we're fine.

So this looks a little too abstract, but it's not the abstract if you now introduce wave functions. So let's do wave functions. So you have a particle, a state of the particle ψ . Time would be random, so I will put just this ψ like that without the bottom line.

And let's look at it. Oh, I want to say one more thing. The x operator acts on the x states to give $x x$. So these are eigenstates of the x operator. We declare them to be eigenstates of the x operator with eigenvalue x . That's their physical interpretation. I probably should have said before.

Now, if we have a ψ as a state or a vector, how do we get the wave function? Well, in this language the wave function, which we call $\psi(x)$, is defined to be the overlap of x with ψ . And that makes sense, because this overlap is a function of

this label here, where the particle is. And therefore, the result is a complex number that is dependent on x .

So this belongs to the complex numbers, because inner products can have complex numbers. Now, I didn't put any complex number here, but when you form states, you can superpose states with complex numbers. So this ψ of x will come out this way.

And now that you are armed with that, you can even think of this in a nicer way. The state ψ is equal to 1 times ψ . And then use the rest of this formula, so this is integral-- $\int dx \langle x | \psi \rangle$. And again, the bracket notation is quite nice, because the bra already meets the ket. This is a number, and this is $\langle x | \psi \rangle$.

This equation has a nice interpretation. It says that the state is a superposition of the basis states, the position states, and the component of your original state along the basis state x is precisely the value of the wave function at x . So the wave function at x is giving you the weight of the state x as it enters into the sum.

So one can compute more things. You will get practice in this type of computations. There are just a limited type of variations that you can do, so it's not that complicated. Basically, you can introduce resolutions of the identity wherever you need them. And if you introduce too many, you waste time, but you typically get the answer anyway.

So it's not too serious. So suppose you want to understand what is the inner product of two states. Put the resolution of the identity in between.

So put ϕ , and then put the integral $\int dx \langle x | \psi \rangle$. Well, the integral goes out, and you get $\langle \phi | \psi \rangle$. And remember, if $\langle x | \psi \rangle$ is $\psi^*(x)$, $\langle \phi | x \rangle$ is the complex conjugate, so it's $\phi^*(x)$.

And you knew that. If you have two wave functions, and you want to compute the overlap, you integrate the complex conjugate of one against the other. So this notation is doing all what you want from this. You want to compute a matrix element of x .

Well, put another resolution of the identity here. So this would be $\int dx \hat{\phi}(\hat{x}) \hat{\psi}(x)$. And then you put $\hat{x} \hat{\psi}(x)$. The \hat{x} on $\hat{\psi}$ is x . That's what this operator does, so you get $\int dx \hat{\phi}(x) x \hat{\psi}(x)$, which is what you expect it to be-- $\int dx \hat{\phi}(x) \hat{x} \hat{\psi}(x)$.

Now we can do exactly the same thing with momentum states. So I don't want to bore you, so I just list the properties-- basis states are momenta where the momenta is real. $\langle p | p' \rangle = \delta(p - p')$. One is the integral $\int dp \delta(p - p')$. And $\hat{p} |p\rangle = p |p\rangle$.

So these are the momentum bases. They're exactly analogous. So all what we've done for x is true. The completeness and normalization work well together, like we checked there, and everything is true.

The only thing that you need to make this more interesting is a relation between the x basis and the p basis. And that's where physics comes in. Anybody can define these two, but then a physical assumption as to what you really mean by momentum is necessary. And what we've said is that the wave function of a particle with momentum p is $e^{i p x / \hbar} / \sqrt{2 \pi \hbar}$ -- convenient normalization, but that was it. That was our physical interpretation of the wave function of a particle with some momentum. And therefore, if this is a wave function, that's $\langle x | p \rangle$. A state of momentum p has this wave function. So we write this.

OK, there are tricks you can do, and please read the notes. But let's do a little computation. Suppose you want to compute what is \hat{p} on $|\psi\rangle$. You could say, well, I don't know why would I want to do something with that? Looks simple enough.

Well, it's simple enough, but you could say I want to see that in terms of wave functions, coordinate space wave functions. Well, if you want to see them in terms of coordinate space wave functions, you have to introduce a complete set of states. So introduce $\int dx \hat{\phi}(x) \hat{\psi}(x)$. Then you have this wave function, and oh, this is sort of known, because it's the complex conjugate of this, so it's $\int dx \hat{\phi}(x) \hat{\psi}(x)$, $\sqrt{2 \pi \hbar} \hat{\psi}(x)$. And this was the Fourier transform-- what

we call the Fourier transform of the wave function. So we can call it $\tilde{\psi}(p)$, just to distinguish it, because we called ψ with x , ψ of x . So if I didn't put a tilde, you might think it's the same functional form, but it's the momentum space wave function.

So here is the wave function in the p basis. It's the Fourier transform of the wave function in the x basis. One last computation, and then we change subjects again.

It's the classic computation that you have now a mixed situation, in which you have the momentum operator states and the coordinate bra. So what is the following expression-- $\hat{p} \psi$? OK.

What is your temptation? Your temptation is to say, look, this is like the momentum operator acting on the wave function in the x basis. It can only be $\hbar/i \, d/dx$ of ψ of x . That's probably what it means.

But the notation is clear enough, so we can check if that is exactly what it is. We can manipulate things already. So let's do it. So for that, I first have to try to get rid of this operator.

Now the only way I know how to get rid of this operator p is because it has eigenstates. So it suggests very strongly that we should introduce momentum states, complete them. So I'll put $\langle p | \hat{p} | p \rangle \psi$.

And now I can evaluate the little-- because \hat{p} and p is little p , or p without the hat. So this is $\langle p | \hat{p} | p \rangle \psi$. Now you can look at that, and think carefully what should you do.

And there's one thing that you can do is look at the equation on top. And this is a way to avoid working very hard. So look at the equation on top-- $x p$ is equal to that.

How do I get a p to multiply this? I can get a p to multiply this $x p$ by doing $\hbar/i \, d/dx$ of $x p$. Because if I see it there, I see that differentiating by d/dx brings down an $i p$ over \hbar . So if I multiply by \hbar/i , I get that.

So let's do this. Now I claim we can take the $\hbar/i \, d/dx$ out of this integral. And

the reason is that first, it's not an x integral. It's a p integral, and nothing else except this factor depends on x .

So I take it out and I want to bring it back, it will only act on this, because this is not x dependent. So you should think of ψ , ψ doesn't have an x dependence. ψ is a state, and here is p -- doesn't have an x dependence? You say no, it does, it looks here.

No, but it doesn't have it, because it's been integrated. It really doesn't have x dependence. So we can take this out. We'll have $\hbar^{-1} \int dx$. And now we have $\int dx p \psi$.

And now by completeness, this is just 1. So this becomes $\int dx \psi$. So $\hbar^{-1} \int dx$ of ψ , which is what we claimed it would be. So this is rigorous-- a rigorous derivation. There's no guessing. We've introduced complete states until you can see how things act. But the moral is here that you shouldn't have to go through this more than once in your life, or practice it. But once you see something like that, you think. You're using x representation, and you're talking about the operator p . It cannot be anything like that.

If you want to practice something different, show that the analogue $p \hat{x} \psi$ is equal to $i \hbar d/dx \psi$. So it's the opposite relation. All right. Questions? Yes.

AUDIENCE: So how's one supposed to-- so what it appears is happening is you're basically taking some state like ψ , and you're basically writing in terms of some basis. And then you're basically using the [INAUDIBLE] coordinates of this thing. But the question is, what does this basis actually look like? Like, what do these vectors-- because if you put them in their own coordinates, they're just infinite.

PROFESSOR: Yup.

AUDIENCE: They're not even delta-- I mean--

PROFESSOR: They are delta functions.

AUDIENCE: [INAUDIBLE]

PROFESSOR: These vectors are delta functions because if you have a state that has this as the position state of a particle, you find the wave function by doing x on it. That's our definition of a wave function. And its infinite.

So there's is not too much one can say about this. If people want to work more mathematically, the more comfortable way, what you do is, instead of taking infinite things, you put everything on a big circle. And then you have a Fourier series and they transform as sums, and everything goes into sums.

But there's no real need. These operations are safe. And we managed to do them, and we're OK with them. Other questions? Yes.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Probably not. You know, infinite bases are delicate. Hilbert spaces are infinite dimensional vector spaces, and they-- not every infinite dimensional space is a Hilbert space. The most important thing of a Hilbert space is this norm, this inner product. But the other important thing is some convergence facts about sequences of vectors that converge to points that are on the space. So it's delicate. Infinite dimensional spaces can be pretty bad. A Banach space is not a Hilbert space. It's more com--

AUDIENCE: [INAUDIBLE]

PROFESSOR: Only for Hilbert spaces, and basically, this problem of a particle in a line, or a particle in three space is sufficiently well known that we're totally comfortable with this somewhat singular operation. So the operator x or the operator p may not be what mathematicians like them to be-- bounded operators in Hilbert spaces. But we know how not to make mistakes with them.

And if you have a very subtle problem, one day you probably have to be more careful. But for the problems we're interested in now, we don't.

So our last topic today is uncertainties and uncertainty relations. I probably won't get

through all of it, but we'll get started. And so we'll have uncertainties. And we will talk about operators, and Hermitian operators.

So here is the question, basically-- if you have a state, we know the result of a measurement of an observable is the eigenvalue of a Hermitian operator. Now, if the state is an eigenstate of the Hermitian operator, you measure the observable, and out comes eigenvalue. And there's no uncertainty in the measured observable, because the measured observable is an eigenvalue and its state is an eigenstate.

The problem arises when the state that you're trying to measure this property is not an eigenstate of the observable. So you know that the interpretation of quantum mechanics is a probabilistic distribution. You sometimes get one thing, sometimes get another thing, depending on the amplitudes of the states to be in those particular eigenstates.

But there's an uncertainty. At this time, you don't know what the measured value will be. So we'll define the uncertainty associated to a Hermitian operator, and we want to define this uncertainty.

So A will be a Hermitian operator. And you were talking about the uncertainty. Now the uncertainty of that operator-- the first thing that you should remember is you can't talk about the uncertainty of the operator unless you give me a state.

So all the formulas we're going to write for uncertainties are uncertainties of operators in some state. So let's call the state ψ . And time will not be relevant, so maybe I should delete the-- well, I'll leave that bar there, just in case.

So we're going to try to define uncertainty. But before we do that, let's try to define another thing-- the expectation value. Well, the expectation value-- you know it. The expectation value of A , and you could put a ψ here if you wish, to remind you that it depends on the state-- is, well, $\psi A \psi$. That's what we call expectation value.

In the inner product notation would be $\psi A \psi$. And one thing you know-- that this thing is real, because the expectation values of Hermitian operators is real. That's

something we reviewed at the beginning of the lecture today.

So now comes the question, what can I do to define an uncertainty of an operator? And an uncertainty-- now we've said already something. I wish to define an uncertainty that is such that the uncertainty is 0 if the state is an eigenstate, and the uncertainty is different from 0 if it's not an eigenstate. In fact, I wish that the uncertainty is 0 if and only if the state is an eigenstate.

So actually, we can achieve that. And in some sense, I think, the most intuitive definition is the one that I will show here. It's that we define the uncertainty, ΔA , and I'll put the ψ here. So this is called the uncertainty of A in the state ψ .

So we'll define it a simple way. What else do we want? We said this should be 0 if and only if the state is an eigenstate. Second, I want this thing to be a real number-- in fact, a positive number. What function do we know in quantum mechanics that can do that magic? Well, it's the norm. The norm function is always real and positive.

So this-- we'll try to set it equal to a norm. So it's the norm of the state A minus the expectation value of A times 1 acting on ψ . This will be our definition of the uncertainty. So it's the norm of this vector.

Now let's look at this. Suppose the norm uncertainty is 0. And if the uncertainty is 0, this vector must be 0. So A minus expectation value of A on ψ is 0. Or $A\psi$ is equal to expectation value of A on ψ .

The 1 doesn't do much. Many people don't write the 1 . I could get tired and stop writing it. You should-- probably it's good manners to write the 1 , but it's not all that necessary. You don't get that confused. If there's an operator and a number here, it must be an identity matrix.

So the uncertainty is 0, the vector is 0, then this is true. Now, you say, well, this equation looks kind of funny, but it says that ψ is an eigenstate of A , because this is a number. It looks a little funny, because we're accustomed to $A\psi = \lambda\psi$, but this is a number.

And in fact, let me show you one thing. If you have $A\psi = \lambda\psi$ -- oh, I should say here that ψ is normalized. If ψ would not be normalized, you change the normalization. You change the uncertainty. So it should be normalized.

And look at this -- if you have $A\psi = \lambda\psi$, do the inner product with ψ . $\langle \psi | A\psi \rangle$ would be equal to λ , because $\langle \psi | \psi \rangle = 1$.

But what is this? This is the expectation value of A . So actually, given our definition, the eigenvalue of some operator on this state is the expectation value of the operator in the state. So back to the argument -- if the uncertainty is 0, the state is an eigenstate. And the eigenvalue happens to be the expectation value -- that is, if the uncertainty is 0.

On the other hand, if you are in an eigenstate, you're here. Then λ is A , and this equation shows that this vector is 0, and therefore you get 0. So you've shown that this norm or this uncertainty is 0, if and only if the state is an eigenstate.

And that's a very powerful statement. The statement that's always known by everybody is that if you have an eigenstate -- yes -- no uncertainty. But if there's no uncertainty, you must have an eigenstate. That's the second part, and uses the fact that the only vector with 0 norm is the zero vector -- a thing that we use over and over again.

So let me make a couple more comments on how you compute this. So that's the uncertainty so far. So the uncertainty vanishes in that case. Now, we can square this equation to find a formula that is perhaps more familiar -- not necessarily more useful, but also good. For computations, it's pretty good -- ΔA of ψ , which is real -- we square it.

Well, the norm square is the inner product of this $A\psi - \langle A \rangle \psi$. Norm squared is the inner product of these two vectors. Now, the thing that we like to do is to move this factor to that side.

How do you move a factor on the first input to the other input? You take the adjoint.

So I should move it with an adjoint. So what do I get? Ψ , and then I get the adjoint and this factor again.

Now, I should put a dagger here, but let me not put it, because A is Hermitian. And moreover, expectation value of A is real. Remember-- so no need for the dagger, so you can put the dagger, and then explain that this is Hermitian and this is real-- or just not put it.

And now look at this. This is a typical calculation. You'll do it many, many times. You just spread out the things. So let me just do it once.

Here you get A^2 minus A expectation value of A minus expectation value of A plus expectation value of A^2 Ψ . So I multiplied everything, but you shouldn't be all that-- I should put a 1 here, probably-- shouldn't worry about this much. This is just a number and an A , a number and an A . The order doesn't matter. These two terms are really the same.

Well, let me go slowly on this once. What is the first term? It's $\Psi A^2 \Psi$, so it's the expectation value of A^2 .

Now, what is this term? Well, you have a number here, which is real. It goes out of whatever you're doing, and you have $\Psi A \Psi$. So this is expectation value of A . And from the leftover $\Psi A \Psi$, you get another expectation value of A . So this is A .

Here the same thing-- the number goes out, and you're left with a $\Psi A \Psi$, which is another expectation value of A , so you get minus A . And you have a plus expectation value of A^2 . And I don't need the i anymore, because the expectation values have been taken.

And this always happens. It's a minus here, a minus here, and a plus here, so there's just one minus at the end of the day. One minus at the end of the day, and a familiar, or famous formula comes out that ΔA on Ψ^2 is equal to the expectation value of A^2 minus expectation value of A squared.

Which shows something quite powerful. This has connections, of course, with statistical mechanics and standard deviations. It's a probabilistic interpretation of this formula, but one fact that this has allowed us to prove is that the expectation value of A squared is always greater or equal than that, because this number is positive, because it is the square of a real positive number. So that's a slightly non-trivial thing, and it's good to know it. And this formula, of course, is very well known.

Now, I'm going to leave a funny geometrical interpretation of the uncertainty. Maybe you will find it illuminating, in some ways turning into pictures all these calculations we've done. I think it actually adds value to it, and I don't think it's very well known, or it's kind of funny, because it must not be very well known. But maybe people don't find it that suggestive. I kind of find it suggestive.

So here's what I want to say geometrically. You have this vector space, and you have a vector ψ . Then you come along, and you add with the operator A . Now the fact that this thing is not an eigenstate means that after you add with A , you don't keep in the same direction. You go in different directions.

So here is $A\psi$. So what can we say here? Well, actually here is this thing. Think of this vector space spanned by ψ . Let's call it $U\psi$. So it's that line there.

You can project this in here, orthogonally. Here is the first claim-- the vector that you get up to here-- this vector-- is nothing else but expectation value of A times ψ . And that makes sense, because it's a number times ψ .

But precisely the orthogonal projection is this. And here, you get an orthogonal vector. We'll call it ψ_{\perp} . And the funny thing about this ψ_{\perp} is that its length is precisely the uncertainty.

So all this, but you could prove-- I'm going to do it. I'm going to show you all these things are true, but it gives you a bit of an insight. you have a vector. A moves you out.

What is the uncertainty is this vertical projection-- vertical thing is the uncertainty. If you're down there, you get nothing. So how do we prove that?

Well, let's construct a projector down to the space U_{ψ} , which is $\psi\psi$. This is a projector, just like any e_1 . e_1 is a projection into the direction of 1. Well, take your first basis vector to be ψ , and that's a projection to ψ .

So let's see what it-- so the projection to ψ . So now let's see what it gives you when it acts on $A\psi$ -- this project acting on $A\psi$ is equal to $\psi\psi A\psi$. And again, the usefulness of bracket notation is kind of nice here.

So what is this? The expectation value of A . So indeed ψ expectation value of A is what you get when you project this down.

So then, the rest is sort of simple. If you take ψ , and subtract from ψ -- well, I'll subtract from ψ , ψ times expectation value of A . I'm sorry, I was saying it wrong. If you think the original vector-- $A\psi$, and subtract from it what we took out, which is ψ times expectation value of A , the projected thing-- this is some vector.

But the main thing is that this vector is orthogonal to ψ . Why? If you take a ψ on the left, this is orthogonal to ψ . And how do you see it? Put the ψ from the left.

And what do you get here? $\psi A\psi$, which is expectation value of A , $\psi\psi$, which is 1, and expectation value A is 0. So this is a vector ψ_{\perp} . And this is, of course, A minus expectation value of A acting on the state ψ .

Well, precisely the norm of ψ_{\perp} is the norm of this, but that's what we defined to be the uncertainty. So indeed, the norm of ψ_{\perp} is ΔA of ψ . So our ideas of projectors and orthogonal projectors allow you to understand better what is the uncertainty-- more pictorially.

You have pictures of vectors, and orthogonal projections, and you want to make the uncertainty 0, you have to push the $A\psi$ into ψ . You have to be an eigenstate, and you're there.

Now, the last thing of-- I'll use the last five minutes to motivate the uncertainty, the famous uncertainty theorem.

And typically, the uncertainty theorem is useful for A and B -- two Hermitian operators. And it relates the uncertainty in A on the state ψ to the uncertainty in B of ψ , saying it must be greater than or equal than some number.

Now, if you look at that, and you think of all the math we've been talking about, you maybe know exactly how you're supposed to prove the uncertainty theorem. Well, what does this remind you of? Cauchy-Schwarz-- Schwarz inequality, I'm sorry-- not Cauchy-Schwarz.

Why? Because for Schwarz inequality, you have norm of u , norm of v is greater than or equal than the norm of the inner product of u and v -- absolute value of the inner product of u and v . Remember, in this thing, this is norm of a vector, this is norm of a vector, and this is value of a scalar. And our uncertainties are norms. So it better be that. That inequality is the only inequality that can possibly give you the answer.

So how would you set this up? You would say define-- as we'll say f equal A minus $\langle A \rangle$ acting on ψ , and g is equal to B minus $\langle B \rangle$ acting on ψ . And then $\langle f, f \rangle$ or $\langle f, f \rangle$ is ΔA squared. $\langle g, g \rangle$ is ΔB squared. And you just need to compute the inner product of f, g , because you need the mixed one. So if you want to have fun, try it. We'll do it next time anyway. All right that's it for today.