PROFESSOR: All right, so our next thing will be getting into the electron spin. And I want to discuss of two things, a little bit of the Dirac equation and the Pauli equation for the electron. We need to understand electron a little better and understand the perturbations, the relativistic corrections so we'll consider the Pauli equation. So the Pauli equation is what we have to do now.

So what is the Pauli equation? It is the first attempt there was to figure out how the Schrodinger equation should be tailored for an electron. So we can also think of the Pauli equation as a baby version of the Dirac equation, which is the complete equation for the electron.

So it all begins at the motivations by considering what is the magnetic moment of an electron? So it's a question that you've been addressing in this course for a while, from Stern-Gerlach experiment and how magnetic fields interact with electron.

So classically and in Gaussian units, if you have a current loop and an area a, the magnetic moment is equal to i times the area vector divided by c.

You can use that to imagine an object that is rotating. And as the object rotates, if its charged, it generates a magnetic moment, and the magnetic moment depends on the amount of rotation of the orbit. So it turns out that, with a little classical argument, you can derive that the magnetic moment is related to the angular momentum by a relation of the form L, so the charge of the object, the mass of the object c, and the angular momentum. And that's perfectly correct classically.

So people thought that quantum mechanically they expect that for an electron or for an elementary particle, you would get q times 2 mc times the spin of the particle. So this would be the intrinsic magnetic moment of an electron that it seems to have times a fudge factor, g that you would have to measure. Because after all, the world is not classical. There's no reason why a classical relation like that would give you the right exact value of the magnetic moment of the electron.

As it turns out, this g happens to be equal to 2. And we get the following. So for electrons g happens to be equal to 2. So mu is equal to 2 times minus e-- the charge of the electron is minus 2 mc, and the spin of the electron is h bar over 2 times sigma, the Pauli matrices. So this thing is minus eh bar over 2 mc sigma mu of the electron.

So if you have a magnetic field, a B external, the Hamiltonian that includes the energetics of the magnetic field interacting with the dipole moment, it's always minus mu dot b. So in this case it would be eh bar over 2 mc sigma dot b. That's the Hamiltonian for an electron in a magnetic field, something you've used many, many times.

The most unpleasant part of this thing is this g equal 2. Why did it turn out to be g equal 2? People knew it was g equal 2. But why? And here we're going to see the beginning of an explanation. In fact, it comes very close to an explanation of that through the Pauli equation.

The Pauli equation is a nice equation you can write, and you can motivate. And it predicts. Once you admit the Pauli equation, it predicts that g would be equal to 2. And that, of course, happens also for the Dirac equation. And Dirac noticed that, and he was very happy about it. So it's quite remarkable how these things showed up, and the G equal to 2 was perfectly natural.

So let's see this Pauli equation first. What is it? Well, when you do the Schrodinger equation for a wave function, you say p squared over 2m-- say a free particle, and you would put the wave function is equal to energies times the wave function. h on the wave function is equal to energy terms the wave function.

So for this electron, you already know it's a two-dimensional Hilbert space, the state space of the up and down. So it's convenient to change this and to say, you know what? I'm going to put here something I'll call a spinor. Chi is a Pauli. It has two things, a chi 1 and a chi 2, and it's a Pauli spinor.

And this is reasonable so far. The spin is defined by 2 degrees of freedom, 2 basis is vectors. So you've done wave functions for spin. And the wave functions for spin have these things, and each themself can be a function of position. So that's perfectly reasonable. I don't think anyone of you is very impressed by this so far.

But here comes the funny thing. In a sense here, there is a 2 by 2 identity matrix sitting here. So Pauli observes the following identity. If you have sigma dot 8 and sigma dot b, it's equal to a dot b times 1 plus i sigma plot a cross b.

Look, these are two vectors, a and b. And if you multiply half this product-- first, this is a dot product. It means sigma 1 times a1 sigma 2 times a2 sigma 3 times a3. So this is a dot product. This product here is a matrix product because this is already a matrix, this is already a matrix, so this is a matrix product.

And this is the result. This comes from properties of the Pauli matrices you already know. In particular, their commutator determines this piece and the anticommutator determines this piece.

So here, if you have, for example, sigma dot p times sigma dot p-- see, these two vectors are the same in this case-- you'll get just p squared times 1 plus 0. Because p times P, those are two vector operators if you wish, but still they commute. So this is equal to 0.

So your p squared can be replaced by sigma dot p sigma dot p. So h, the Hamiltonian-- that is p squared over 2 m times the identity matrix-- can be perhaps better thought as sigma dot p times sigma dot p divided by 2m.

So this is the first step. You haven't done really much, but you've rewritten the Hamiltonian with a unit matrix here perhaps in a somewhat provocative way. In quantum mechanics, when we couple to electromagnetism, there are simple changes we have to do. And we will study that in detail in about three weeks, but today I will just bench on what you're supposed to do in order to couple to electromagnetism.

In order to couple to electromagnetism, you're supposed to change p wherever you see by an object you can call pi. It's some sort of more canonical momentum in which it's got to p minus q over c. The vector potential is a function of x. Well, people write it like this, qp minus q ac. So you're supposed to do this replacement. When you're dealing with a particle moving in some electromagnetic field, that's the change you must do. We will study that in detail.

But I think an obvious question at this moment is the following. You say, well, I'm in quantum mechanics, and I work with p and x. Those are my opera-- what is a? is it an operator? Is it a vector? Is it what? p is an operator, but what is a? You should think of this vector potential. In general, this vector potential will depend on the position. So if you put a magnetic field, you require a vector potential. It has some position dependent.

So actually, this a here should be thought as a of x, the same way as when you have the potential that depends on x. In quantum mechanics, we just think of that x as an operator. So the a is an operator because x is an operator, and a has x dependents.

So the Pauli Hamiltonian, h Pauli is nothing else but sigma dot pi times sigma dot pi over 2 m.

Because we said p must be replaced by pi, so this is the Pauli Hamiltonian. And it's equal to 1 over 2m pi squared pi dot pi times 1 plus i sigma dotted with pi cross pi.

OK, here is the computation we need to do. What is pi cross pi? You know what pi is. It's given by this thing. So how much is pi cross pi?

Well, it takes a little computation. I'll tell you what you have to do. It's a very interesting computation. It's small. It reminds you of this computation in angular momentum, l cross l. Maybe you've written the algebra of angular momentum in this language. It's equal to h bar l. That's your computation in relations of angular momentum written like that. So pi cross pi, the kth component is epsilon ijk pi i pi j or 1/2 epsilon ijk pi I commutator with pi j. This last step, it can be explained by writing out the commutator, which is pi i pi j minus pi j pi i. But with this epsilon, those two terms are the same, and it becomes this.

So how much is this thing? I'll leave it for you to do it. It's simple thing. You have to commute these things, and you must think of p as derivatives. So this pi i pi j is i h bar q over c di a j minus dj a i.

You can see it coming. You see, you have a commutator of two factors like that. The a with a will commute. The p with p will commute. The cross products won't, but that's just derivatives. So therefore, you get the derivatives of a antisymmetrized. And the derivatives of a antisymmetrized is the curl of a. And the curl of a is the magnetic field. So that's why this happens.

So at the end of the day, when you'll finish this computation, get that pi cross pi is just i q i h bar q over c times the magnetic field. Pretty nice equation.

So the Pauli Hamiltonian includes h Pauli. It includes this term, which is i sigma over 2m dotted with i hQ over cb plus the other terms. I'm just looking at this term which has the magnetic field.

And therefore, look at this. i with i is minus 1, but q is minus e. So this is eh bar over 2 mc sigma dot b, which is here, which came from g equal 2. Nowhere I had to say there that g is equal to 2. It came out of this calculation. The Pauli Hamiltonian knew of this. And therefore, it's a great progress, that Pauli Hamiltonian, but it suggests to us that we can do things still even better, as Dirac did.