PROFESSOR: OK. Let's turn then to the Dirac equation and motivated, basically. Through the Dirac equation is the simplest way to do perturbation theory for the hydrogen atom. It helps us derive what we should think of perturbation. So I'll discuss that quickly. And there may be some of that done in recitation. So we're going to discuss now the Dirac equation.

So the Dirac equation begins with the observation that we have E squared minus p squared c squared for a free particle is m squared c to the fourth. This dispersion relation that rates E and p for any particle. So if you wanted to describe the dynamics of a relativistic particle, you could say, look, the energy is the square root of p squared c squared plus m squared c to the fourth. Therefore, I should take the Hamiltonian to be that thing. I should take H' to be that. And work with a Schrodinger equation that has this H with the square root.

Nobody does that, of course. But you can do a little bit with it. If you notice that this is equal to mc squared square root of 1 plus p squared over m squared c squared. And then expand. In small velocities where this is less than 1, you get 1 plus 1/2 p squared over m squared c squared minus 1/8 of this term squared. So p squared p squared over m to the fourth c to the fourth.

And then what do you get? mc squared plus 1/2 2m p squared. That's very nice. Apart from the rest energy, you get the p squared over 2m that you would expect. And the next correction is minus 18 p squared p squared over m cube c squared plus dot dot dot.

You know, that's one way you could treat the Hamiltonian. But it's very complicated. Don't treat the Hamiltonian too seriously because the Schrodinger equation would have infinite number of spatial derivatives. p to the fourth, p to the sixth, p to the eighth. Never going to get that understood. But in terms of perturbation theory, we can think of p squared over 2m, your original Hamiltonian. And here is a relativistic correction that is small.

But Dirac was puzzled by that square root. He basically looked at it and I think he had sort of mathematical inspiration in here. He looked at this equation and that square root and he said, how much would I wish I could take that square root. What can I do to take that square root?

So he said, well, I could take that square root. This has p squared so c squared plus m squared c to the fourth. I could take that square root if that would be equal to something squared. Which it's not because it's missing the cross product of the right amount. And it's not

there.

But if it would be there, the cross product, it would be a linear function of p and mc squared. So he says, OK, let me put a linear function of p. And the most general linear function of p is obtained, I want the c squared, so c, by doing the dot product of a vector with p, a constant vector.

A constant vector times p is the most general linear function of p. And here I'll output another constant, mc squared. And I hope I can take the square root because there will exist some constants and it somehow will work. As of now, it cannot work because it's always across product. But just follow these dots.

And then alpha dot p. This is equal to c alpha 1 p1 plus c alpha 2 p2 plus c alpha 3 p3 plus beta mc squared. So let's list all the things that should happen for this to work out. Well, this square includes the square of this, the square of this, the square of this, and the square of this. And those are what we want. So we should have that alpha 1 squared is equal to alpha 2 squared is equal to alpha 3 squared is equal to beta squared is equal to 1.

If that happens, well, this term will give you c squared p1 squared, c squared p2 squared, c squared p3 squared, and m squared c to the fourth, and I get everything to work. But the cross products must also work. So for example, the cross product of p1 with p2 should vanish.

And that, well, if you ride these two factors like that here you would say, oh, what they need now is that alpha i times alpha j plus alpha j times alpha i should be 0 for all i different from j. So for example, alpha 1, alpha 2 plus alpha 2 alpha 1, when you do the product, should vanish. And I kept the order of these things there. When I'm starting to think that maybe numbers is not going to work. So i difference from j. This must happen too for this to be equal.

And moreover for all i alpha i beta. There shouldn't be cross products between these ones and this term either. Plus beta alpha i is equal to 0. And that's all that should happen. If that happens, we can take the square root.

But if these things are numbers, well, this could be phases or 1 or minus 1. And that's never going to work. This is twice this product. So [INAUDIBLE] OK, they're matrices. But that's not bad, if they're matrices, we'll have like spinors, like Pauli was doing. And we'll work on spinors. But it turns out that this one, the simplest solution is something with 4 by 4 matrices.

So the solution is alpha is 0 sigma sigma 0. And beta is 1 minus 100. This is the simplest solution, 4 by 4 matrices. So that Dirac Hamiltonian, H Dirac, is equal to the energy. And the energy was the square root of this. So it's just this thing. So it's c alpha dot p plus b mc squared. That's our Dirac Hamiltonian.

But we need to deal with spinors that now are four dimensional. These matrices, alpha and beta, are four dimensional because the segments that are inside are two dimensional. And these are 2 by 2 matrices.

So when you write the Dirac equation as i H bar d psi dt equal H Dirac psi, psi is a four component thing, a chi and a phi. And what people discover is that chi behaves like the Pauli spinor. And this one is a small thing, a small correction.

So there is a whole analysis of this system in which, in order to put the magnetic field, you can put in here for the Dirac Hamiltonian, H Dirac, coupled to electromagetism has a c alpha p plus e over ca plus beta mc squared. And you still have to add the potential, V of r, that comes from [INAUDIBLE] minus e squared over r. It's the value of the electron charge that is the potential of r. The scalar potential phi of r.

So this is the Dirac Hamiltonian. And you have these things. And this is something you can read if you're interested in [? Shankar. ?] There is a derivation of the Pauli equation for the electron. Which is the spinors that you are familiar with by eliminating in a recursive expansion phi. And it may be that Professor Metlitski is going to go through that in recitation as well.

So what happens? What you get is that you get an H chi equal E chi. And H is what? Here is the grand prize. From the Dirac equation, we can rewrite the Hamiltonian of the hydrogen atom in a more accurate way, a more complete Hamiltonian. And it has p squared over 2m plus V of r, which is what we've called H0.

It's there. It's the first term. That's what you would expect. Then there is that term that we anticipated. Some relativistic correction here. So it comes like minus p to the fourth over 8m cubed c squared. And we call this delta H relativistic.

And then you get this term that corresponds to spin orbit coupling that also shows up. 2m squared c squared 1 over r dv dr. You've studied this thing and you gave a heuristic explanation for it. That's another term. It's kind of difficult to get the right value by a simple argument. You always get it off by a factor of 2 that is associated to Thomas precession. But

here it comes out directly with the right number. This is called spin orbit coupling. Delta H of spin orbit.

And there's one more term. Plus H squared to this [INAUDIBLE] 8 m squared c squared Laplacian of V. Which is called of this V of r. It's called the Darwin term. Delta H Darwin. Not the biologist. Some Darwin.

So the job is set for us. We have to do perturbation theory. Now you remember that all the terms associated with H0 were of the form energies with alpha squared mc squared. This was like H0. If you look at the units of all of these terms, we looked at them in the notes. Delta H. They all go like alpha to the fourth mc squared.

So there is a difference of alpha squared there. 1/19000 smaller. That is the fine structure of the hydrogen atom. And our task next time will be to compute the effect of this and see what happens with all the levels of the hydrogen atom. This is our [? final ?] structure. So we'll do that next time.