Classical Mechanics III (8.09) Fall 2014
Assignment 10

Massachusetts Institute of Technology
Physics Department *Due Fri. December 5, 2014*
Fri. November 28, 2014 *6:00pm*

Announcements

This week we continue our study of nonlinear dynamics and chaos, including bifurcations and limit cycles, followed by chaos in maps, fractals, and strange attractors.

Reading Assignment

- Read the posted sections from Chapter 3 of **Strogatz** on Bifurcations.
- Read the posted sections from **Strogatz** on fixed points in two dimensions and limit cycles: 5.1-5.3, 6.1-6.5, 7.1-7.3, 8.1-8.2, and 8.4 (also have a look at 5.3 on love affairs, and 6.7 for pendulum phase space on a cylinder if you like).
- Read **Goldstein** section 11.8 on the logistic map, and section 11.9 on fractals.

Problem Set 10

These 5 problems are on nonlinear dynamics and chaos.

1. **Classifying Fixed Points** [8 points]

   Find all fixed points of the following system:

   $$\dot{x} = x(4 + y - x^2)\,, \qquad \dot{y} = y(x-1)$$

   Determine their stability and type, and use this information to sketch trajectories for this system in the (x, y) plane. Find the corresponding eigenvectors for cases where this gives useful information.

2. **Bead on a Rotating Hoop** [15 points]

   Consider a bead of mass m on a hoop of radius a with friction coefficient $\beta > 0$. The hoop is vertical and rotates about the z-axis with constant angular velocity ω_0, so that the bead's equation of motion is

   $$m\,a\,\ddot{\theta} = -\beta\,\dot{\theta} + ma\omega_0^2\,\sin\theta\Big(\cos\theta - \frac{g}{a\omega_0^2}\Big)$$

   We analyzed this problem in lecture for extreme overdamping where we neglected the $\ddot{\theta}$ term. Here you will analyze the problem in two dimensions.

   (a) [2 points] Show that by suitable changes of variable the equations of motion can be written in the dimensionless form

   $$\dot{\theta} = w\,, \qquad \dot{w} = \sin\theta\Big(\cos\theta - \frac{1}{\gamma}\Big) - bw$$

   with $\gamma > 0$ and $b > 0$. Can you identify a symmetry of this system involving both variables?

   (b) [5 points] What are the fixed points if $b = 0$? If $b \neq 0$? For both of these cases, classify the stability and type of all the fixed points with a linear analysis (only).

   (c) [5 points] Consider the undamped case $b = 0$. Demonstrate that the system is conservative and find a conserved quantity $H(\theta, w)$. Is your conserved quantity energy? Why or why not? By plotting curves of constant H draw the phase space trajectories in (θ, w). I suggest using mathematica or a similar program to make this plot. Be sure to also label the fixed points.

   (d) [3 points] Consider now $b = 1$ and $\gamma = 2$. Sketch a trajectory that starts near each stable fixed point. Sketch one trajectory that has an initial $w(t = 0)$ that is large enough for the bead to go over the top of the hoop. [You could use NDSolve in mathematica to numerically solve the equations and accurately plot various trajectories, but you are not required to do so. If you choose to do this, try $b = 1/2$ too.]

3. **Chaos in an Undamped Nonlinear Oscillator** [10 points]

You may have wondered if the damping was important in our discussion of chaos for the driven nonlinear oscillator. Consider the forced nonlinear oscillator without damping (quality $q = \infty$), which has

$$\dot{\theta} = w \ , \qquad \dot{w} = -\sin\theta - a\cos\phi \ , \qquad \dot{\phi} = w_D$$

Start with the original mathematica code for the forced nonlinear oscillator from the website that you considered on problem set #9 (note that our $w_D = \omega$ in the notebook). You can turn off damping by using the slide bar to set "$Q = 0$" (you may have noticed last week that there is an IF command present in the code that uses $Q = 0$ to set no damping). Take $w_D = 0.7$.

(a) [4 points] Use the code to create a bifurcation plot showing at least $0.1 < a < 2$ (choose 300 intervals to get high resolution). Identify a value of a that corresponds to a periodic window in between chaotic regions. Show Poincaré sections to prove it.

(b) [6 points] Examine the behavior more closely for small a's. Start by looking closely at $0 < a < 0.1$ (Make a bifurcation plot. Also consider other plots.) Are their chaotic values? For what value in $0 < a < 1$ does chaos first appear? Be sure to test and justify your answer.

4. **Bifurcation of a Limit Cycle and Fixed Point** [12 points]

Consider a system governed by the equation

$$\ddot{x} + a\,\dot{x}\,(x^2 + \dot{x}^2 - 1) + x = 0$$

(a) [4 points] Let $\dot{x} = w$ and form first order equations. Show that the system has a circular limit cycle for $a \neq 0$ and find its amplitude and period. (You can demonstrate that it is isolated using your results from part (c), so you should comment on this either here or in part (c).)

(b) [4 points] Find and classify all the fixed points for $a > 0$, $a < 0$, and $a = 0$. Determine the stability in all cases. Determine the type of fixed point, but for cases which are on a borderline do not bother going beyond the linear analysis. Based on fixed point stability and the behavior of $\dot{w}$ outside the limit cycle, make a proposal for when the limit cycle is stable, unstable, or half-stable.

(c) [4 points] Change variables to polar coordinates, $x = r\cos\theta$ and $w = r\sin\theta$. Derive a first order equation for $\dot{r}$ and use it to prove your claim from (b). Thus determine a bifurcation point for the limit cycle and fixed point. What would be a reasonable name for this bifurcation?

5. **Chaos in Maps** [15 points]

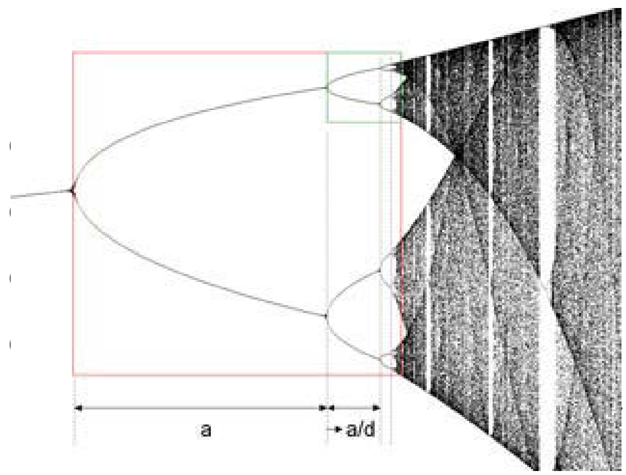
We have seen that chaos from 1st order differential equations requires 3 variables and nonlinearity. Instead consider a discrete set of points $\{x_n\}$ determined by a *map*, $x_{n+1} = f(x_n)$. Here chaos can occur for a nonlinear function $f(x)$ with only one variable x. You can picture why this might be the case by recalling that for the driven damped nonlinear oscillator we saw chaos when we plotted a discrete set of points ω_n versus the control parameter a (with values obtained from Poincaré sections).

In this problem you will numerically explore some features of one dimensional maps by making use of the mathematica code that is available on the course website. (Print any plots and attach them to your pset.) We will also discuss maps in lecture, but this problem is designed so that you can solve it without use of any lecture material.

Perhaps the simplest map that exhibits chaos is the Logistic Map

$$x_{n+1} = r\, x_n\, (1 - x_n)$$

where r is a fixed control parameter. The figure shows its bifurcation plot obtained by discarding x_0 to x_{299} and plotting x_{300} to x_{600} for many different r values. Code for this plot is on the website.



Just like differential equations, maps contain attractors, so after transients have died out the values will be independent of the initial condition x_0. We see a period doubling road to chaos (compare this to the bifurcation plot for our driven damped nonlinear oscillator).

Lets start with a few problems on the Logistic Map:

(a) [3 points] To test the sensitivity to initial conditions compute the set of points $\{x_n\}$ and $\{x'_n\}$, starting with two nearby values x_0 and x'_0 respectively. Consider one value of r where the map becomes periodic, and one value of r where it is chaotic. For both r values plot the difference of your lists $\{x_n - x'_n\}$, being sure to plot to large enough n that you see the expected outcome. For the chaotic case, how close do you have to take $x_0 - x'_0$ if you want to ensure that $|x_{15} - x'_{15}| < 10^{-6}$?

(b) [4 points] By adjusting the plot region (and perhaps using mathematica's "get coordinates" feature obtained by a right-click of the mouse) find the first four x values where bifurcations occur. Call them $a_{1,2,3,4}$ and compute two values for $(a_n - a_{n-1})/(a_{n+1} - a_n)$, which should agree (say with at least two significant digits). This is a small n approximation to Feigenbaum's number which is the constant obtained by taking the $n \to \infty$ limit. It is a universal constant characterizing the period doubling route to chaos in many chaotic systems. (Your task is much simpler than it was for Feigenbaum, who did his computations with a hand calculator.)

(c) [3 points] Show that in the region after the first bifurcation, but before the second bifurcation, that the two values of x for the attractor satisfy

$$r^3x^2(2-x) - r^2(r+1)x + (r^2-1) = 0$$

Finally lets explore bifurcation plots for a few other maps.

(d) [5 points] Use the mathematica code to make bifurcation plots for the following three maps. All your plots should exhibit period doubling. Two of these maps are chaotic, and for them you should pick a range of r so that your plot includes both chaotic and non-chaotic regions.

$$\text{(i) } x_{n+1} = r\cos x_n\,, \qquad \text{(ii) } x_{n+1} = rx_n - x_n^3\,, \qquad \text{(iii) } x_{n+1} = \exp(-rx_n)$$