Massachusetts Institute of Technology

8.223, Classical Mechanics II

Exercises 3

23. Verify the Virial Theorem for a one dimensional simple harmonic oscillator by direct calculation, i.e. compute $T(t)$ and $U(t)$ and find their averages over one cycle.

24. Compute the cross-section for back-scattering off a fixed impenetrable sphere of radius R (i.e., $U = 0$ for $r > R$, and $U = \infty$ for $r \leq R$, and scattering angle $|\theta| > \pi/2$).

25. Show that a solution to

$$\ddot{x} + \omega_o^2 x = \frac{F}{m}\cos(\omega t + \theta) \qquad (1)$$

for the case of resonant driving ($\omega_o = \omega$) is $x(t) = a_1 \cos(\omega_o t + \phi) + a_2\, t \sin(\omega_o t + \theta)$. Find the constants a_1 and ϕ for the initial conditions $x(0) = 0$ and $\dot{x}(0) = v_o$.

26. ($\times 2$) Small Oscillations: For the system in problem 16 (pset 2), compute the angular frequency ω for small oscillations about (stable) equilibrium.

27. Review of damped undriven and driven one dimensional harmonic oscillators

a $\times 2$) The equation of motion for an undriven harmonic oscillator is

$$m\ddot{x} = -\lambda\dot{x} - kx.$$

Use a trial solution $x(t) = e^{-ct}$, substitute in the equation, and show that there are three solutions depending on whether the oscillator is under damped, critically damped or over damped:

i) $x(t) = e^{-\Lambda t}\left[A \sin \omega t + B \cos \omega t\right]$
ii) $x(t) = e^{-\Lambda t}\left[At + B\right]$
iii) $x(t) = Ae^{\Lambda_1 t} + Be^{\Lambda_2 t}$

Find the values of the Λ's and ω for each case, in terms of m, λ and k. (Note, k is the "spring constant" as in the conservative potential $U(x) = \frac{1}{2}kx^2$, λ is the damping coefficient, and m the mass.)

b) A driven damped simple harmonic oscillator obeys the equation

$$m\ddot{x} = -\lambda\dot{x} - kx + C\sin\omega t$$

and its solution has the form $x(t) = x_I(t) + x_{II}(t)$ where $x_I(t)$ is the transient solution and has the form of the solution in part a). Show that $x_{II}(t)$, the steady state solution, has the form

$$x_{II}(t) = \frac{D}{\sqrt{(\omega^2 - \omega_o^2)^2 + \Gamma^2}} \sin(\omega t + \phi)$$

and find ω_o, D, Γ and ϕ in terms of the constants describing the properties of the oscillator (m, λ and k) and the drive (C and ω).

28. A driven oscillator is described by

$$\ddot{x} + \omega_o^2 x = \frac{F}{m}\cos(\gamma t + \alpha). \tag{2}$$

We found that the solution off resonance is

$$x(t) = B\cos(\omega_o t + \beta) + \frac{F/m}{\omega_o^2 - \gamma^2}\cos(\gamma t + \alpha).$$

which we can rearrange to

$$x(t) = C\cos(\omega_o t + \kappa) + \frac{F/m}{\omega_o^2 - \gamma^2}\left(\cos(\gamma t + \alpha) - \cos(\omega_o t + \alpha)\right).$$

with new constants C and κ.

a) If the oscillator is driven close to the natural frequency ω_o, we can write $\omega_o = \gamma + \epsilon$ with $\epsilon \ll \omega_o$. Keeping terms only linear in ϵ (i.e. set any ϵ with higher power to zero), show that we can write

$$x(t) \;=\; C\cos(\omega_o t + \kappa) + \frac{F/m}{2\omega_o\epsilon}(\cos(\omega_o t + \alpha - \epsilon t) - \cos(\omega_o t + \alpha)) \tag{3}$$

b) Show that this evolves to the on resonance solution (LL 22.5) for $\epsilon \to 0$. Note: you may carry out the calculation using trigonometric identities or complex notation.

Note: to compare with LL 22.5, convert the above as follows:

$$C \to a, \quad F \to f, \quad \omega_0 \to \omega, \quad \kappa \to \alpha, \quad \alpha \to \beta$$

29. ($\times 2$) Determine the positions of stable equilibrium of a pendulum whose point of support, x_s, oscillates horizontally with high frequency: $x_s = a\cos(\gamma t)$, with $\gamma \gg \sqrt{g/l}$ (i.e., a horizontal Kapitza pendulum).

30. OPTIONAL: We can write the solution to a simple harmonic oscillator as

$$\begin{aligned} x(t) &= x_o \cos\omega_o t + \frac{v_o}{\omega_o}\sin\omega_o t \\ &= x_1(t, x_o) + x_2(t, v_o). \end{aligned}$$

After a time Δt, the solution will be $x(t + \Delta t)$ which we may write

$$\begin{aligned} x_1(t + \Delta t, x_o) &= a x_1(t, x_o) + b x_2(t, v_o) \\ x_2(t + \Delta t, v_o) &= c x_1(t, x_o) + d x_2(t, v_o). \end{aligned}$$

Find a, b, c and d.

31. OPTIONAL: We can use a, b, c and d from the previous problem to make the matrix M such that $\vec{x}(t + \Delta t) = M\vec{x}(t)$. Find the eigenvalues of M. Take $\Delta t = 4\pi/\omega_o$ and find the eigenvectors.