# Lecture 6

For next time

1. read LL 13-15
2. do pset 15-16

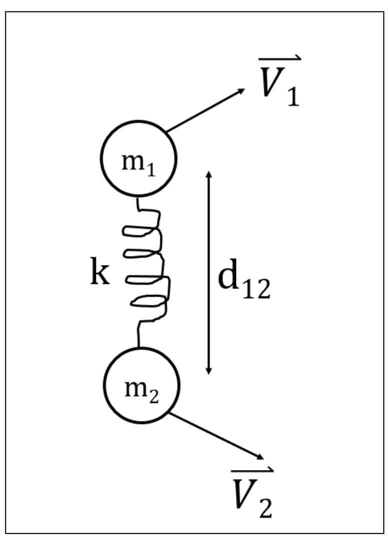
Last time we talked about several instances in which a symmetry of space or time resulted in a conserved quantity; conservation of energy, momentum, and angular momentum come from homogeneous time and space, and isotropic space. Each of these is an example of

Noether's theorem which states that "every differentiable symmetry of the action of a physical system has a corresponding conservation law."

Today we will go through an example in which we use the things we talked about last time to find the equations of motion of a diatomic molecule.

## 1 Example Diatomic Molecule

We will start with the Lagrangian of a simple system of 2 equal masses connected by a spring. (The spring is massless and acts only in 1 DOF; longitudinal).

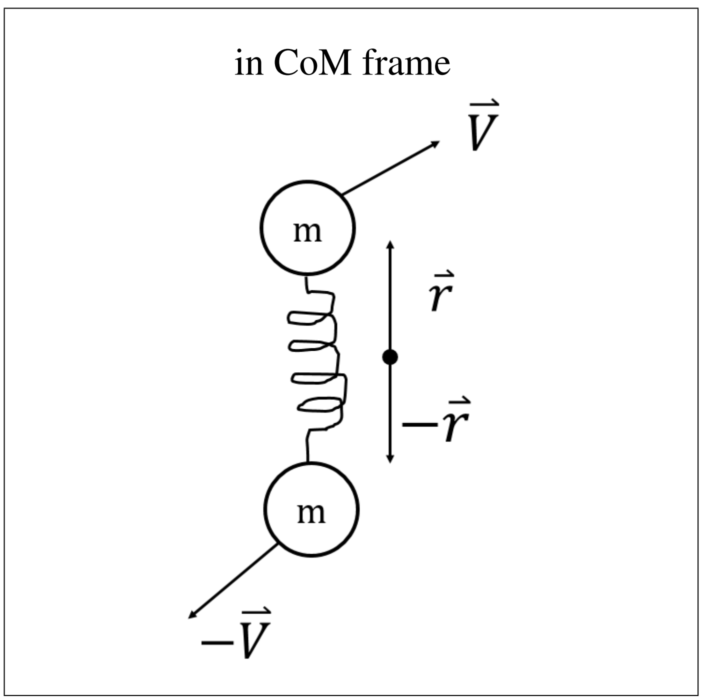


We start with the usual declaration of our intent to find the EoM; the Lagrangian:

$$U(\vec{r_1},\vec{r_2}) = \frac{1}{2}k|\vec{r_1} - \vec{r_2}|^2$$
$$L = \frac{1}{2}m(v_1^2 + v_2^2) - U(\vec{r_1},\vec{r_2})$$

where the masses have velocities $\vec{v}_1$ and $\vec{v}_2$. Our problem is to eliminate at least 10 initial conditions (and 5 corresponding EoM) in favor of constants of the motion, and then find the EoM for the distance between the particles.

First, we move to CoM frame




$$\vec{R}_{CM} = \frac{m\vec{r}_1 + m\vec{r}_2}{2m} = \frac{\vec{r}_1 + \vec{r}_2}{2} \quad , \quad \vec{v}_{CM} = \frac{\vec{v}_1 + \vec{v}_2}{2}$$

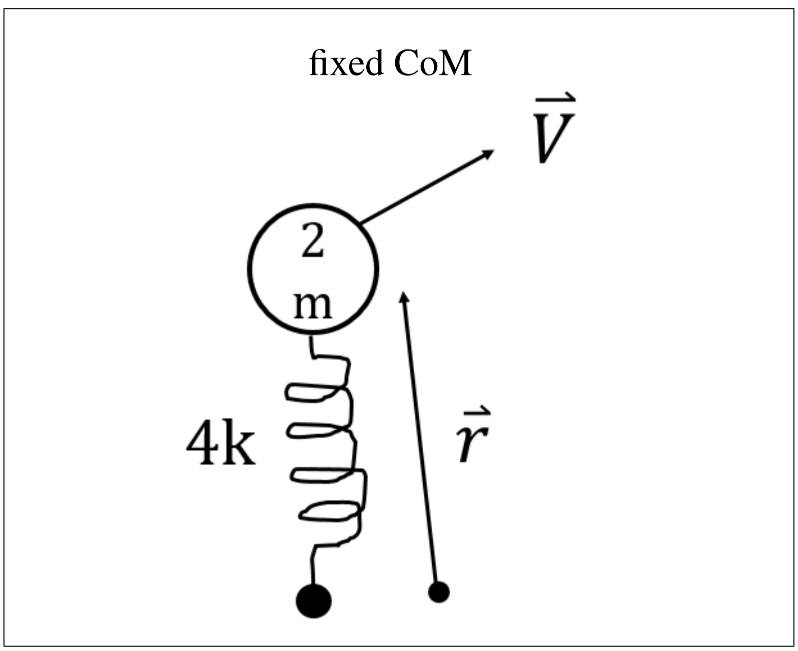
Where the particles' positions and velocities in the CoM frame are

$$\begin{aligned}\vec{r}_1' &= \vec{r}_1 - \vec{R}_{CM} = -\vec{r}_2' \equiv \vec{r} \\ \vec{v}_1' &= \vec{v}_1 - \vec{v}_{CM} = \frac{\vec{v}_1 - \vec{v}_2}{2} = -\vec{v}_2' \equiv \vec{v}\end{aligned}$$

and the internal energy and angular momentum are

$$\begin{aligned}E_i &= \frac{1}{2}Mv^2 + \frac{1}{2}(4k)r^2 \\ \vec{L}_i &= M\,\vec{r} \times \vec{v} \ , \quad M = 2m\end{aligned}$$

(Of course, $\vec{P} = 0$ in the CoM frame!)




In the CoM frame, both masses move symmetrically about the CoM which appears fixed. The new Lagrangian, now in spherical coordinates with the origin at the CoM is

$$\begin{aligned} L &= \frac{1}{2}M(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2) - 2kr^2 \\ F_r &= \frac{\partial L}{\partial r} = Mr(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2) - 4kr \\ p_r &= \frac{\partial L}{\partial \dot{r}} = M\dot{r} \end{aligned}$$

With the CoM at the origin, our problem has some easy symmetries we can take advantage of. The potential depends only on $|\vec{r}|$, so AM is conserved:

$$\begin{aligned} |\vec{L}|^2 &= M^2r^4(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2) = \text{const} \\ & \quad \text{let's define } \lambda = \frac{\left|\vec{L}_i\right|^2}{M} \\ \Rightarrow F_r &= \frac{\lambda}{r^3} - 4kr \\ \Rightarrow M\ddot{r} &= \frac{\lambda}{r^3} - 4kr \quad \text{EoM for r} \end{aligned}$$

In the end, we have a fairly simple EoM. This is entirely due to:

1. conservation of momentum, which makes the CoM coordinate choice special (i.e. CoM is a fixed point)

2. conservation of angular momentum, which removes all of the angles from the EoM for $r$

To turn this into a trajectory, we need only add the initial radius and radial velocity (and solve the 2nd order ODE).

Another way to find an EoM for $r$ is to by-pass the Lagrangian and use energy conservation:

$$\begin{aligned} E = E_i &= \frac{1}{2}M\left(\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)\right) + 2kr^2 \\ &= \frac{1}{2}M\dot{r}^2 + \frac{\lambda}{2r^2} + 2kr^2 \\ \Rightarrow \frac{1}{2}M\dot{r}^2 &= E_i - \left(\frac{\lambda}{2r^2} + 2kr^2\right) \end{aligned}$$

Which is not a nice ODE, but it is first-order. This means that the initial velocity information is already contained in the constant $E_i$, so you only need $r$ at $t = 0$ to solve. This approach is not general, but if you have only one variable remaining to integrate, it can work.

*In general, initial conditions are trivially "constants of motion", but exchanging them for conserved quantities can significantly simplify the Equations of Motion.*

## 2 Mechanical Similarity

This curious little chapter of LL turns out to be very interesting if only for the fact that the concept it explores led to the discovery of dark matter.

The idea is that the form of the E-L equations can tell us something about motion in a particular potential even before we solve them to find the EoM.

Mechanical Similarity

$$\begin{aligned} U(\alpha r_1, \alpha r_2) &= \alpha^k U(r_1, r_2) \\ \text{e.g. } U(r_1, r_2) &= \frac{1}{2}a|\vec{r_1} - \vec{r_2}|^2 \\ \Rightarrow U(\alpha\vec{r_1}, \alpha\vec{r_2}) &= \alpha^2 U(\vec{r_1}, \vec{r_2}) \Rightarrow k = 2 \end{aligned}$$

This just says "If you double the size of the system, you quadruple the potential energy". What does this do to the EoM?

$$
\begin{aligned}
q' = \alpha q \quad , \quad t' = \beta t \;\Rightarrow\; \dot{q}' &= \frac{\mathrm{d}}{\mathrm{d}t'}q' = \frac{1}{\beta}\frac{\mathrm{d}}{\mathrm{d}t}(\alpha q) = \frac{\alpha}{\beta}\dot{q} \\
\Rightarrow L' &= L(q', \dot{q}') = L(\alpha q, \frac{\alpha}{\beta}\dot{q}) \\
&= \frac{1}{2}m\dot{q}'^2 + U(q') = \frac{1}{2}m\left(\frac{\alpha}{\beta}\dot{q}\right)^2 + \alpha^k U(q) \\
&= \alpha^k L \text{ if } \frac{\alpha^2}{\beta^2} = \alpha^k \;\Rightarrow\; \beta = \alpha^{1-\frac{k}{2}}
\end{aligned}
$$

Multiplying L by a constant doesn't change the EoM, so with this condition on $\beta$ (scaling of time), the EoMs are equal. A few important examples of this come from the harmonic oscillator and orbits in a gravitational potential. First the Harmonic Oscillator:

Harmonic Oscillator Potential

$$
\begin{aligned}
&\text{Period } \tau, \text{Amplitude } A, U(x) \propto x^2 \Rightarrow k = 2 \\
&A' = \alpha A,\; \tau' = \beta\tau,\; k = 2 \Rightarrow \beta = \alpha^{1-\frac{2}{2}} = 1 \\
&\qquad\qquad \Rightarrow \tau \text{ is independent of A}
\end{aligned}
$$

This is the founding principle of all kinds of tic-toc clocks. (Even digital clocks use crystals, which have a $k = 2$ mechanical resonance).

For gravity, this looks like:

Gravitational Potential

$$
\begin{aligned}
&\text{Period } \tau, \text{Radius } R, U(r) \propto 1/r \Rightarrow k = -1 \\
&R' = \alpha R,\; \tau' = \beta\tau,\; k = -1 \Rightarrow \beta = \alpha^{1-\frac{-1}{2}} = \alpha^{3/2}
\end{aligned}
$$

which is related to Kepler's third law, but we will get back to that in a few lectures.

We can also take a different approach to scaling... let's look at how the kinetic energy scales relative to potential energy given a $r^k$ potential. There are 2 tricks we need for this:

$$
\begin{aligned}
\text{for } U(q) &= aq^k \Rightarrow q\,\frac{\partial U}{\partial q} = q\,akq^{k-1} = kU \\
\text{and } \frac{\mathrm{d}}{\mathrm{d}t}(pq) &= \dot{p}q + \dot{q}p \\
T = \frac{1}{2}mv^2 &= \frac{1}{2}\dot{q}p \;\;, \;\; \dot{p} = F = -\frac{\partial U}{\partial q} \\
\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}(pq) &= -q\frac{\partial U}{\partial q} + 2T = -kU + 2T
\end{aligned}
$$

The dramatic part happens when we average this over any finite/bounded trajectory (think orbits or oscillations in any potential well).

$$
\begin{aligned}
\overline{f(t)} = \lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau f(t)dt \\
\Rightarrow \overline{\frac{\mathrm{d}}{\mathrm{d}t}(pq)} = \lim_{\tau\to\infty}\frac{1}{\tau}\left(pq|_{t=0}^{t=\tau}\right) = 0 \\
\Rightarrow 2\overline{T} = k\overline{U} \text{ for } U \propto r^2 \Rightarrow \overline{T} = \overline{U} \\
\text{(harmonic oscillator)} \\
\text{(gravity) } U \propto \frac{1}{r} \;\Rightarrow 2\overline{T} = -\overline{U} \;\;\text{“Virial Theorem”}
\end{aligned}
$$

What this means for astronomy is that if you can measure the velocities of stars in a galaxy (actually, the velocity dispersion), you can measure $\overline{T}$. From that you can compute $\overline{U}$. You can then compare that to the potential you would expect from the stars you see. And you would discover that they don't match at all! Dark matter! (factor $\approx 4$). Simply put, stars in galaxies move too fast for the gravity of the stars. If there were nothing else, the galaxy would fly apart!

*For next time*

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8.223 Classical Mechanics II
January IAP 2017