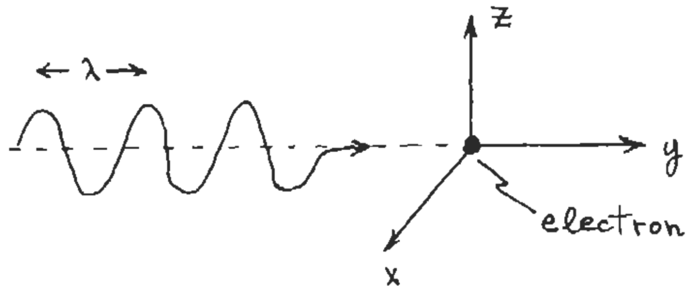
8.282

Thomson and Rayleigh Scattering

Consider a free electron at $y=0$ in the presence of a passing electromagnetic wave of frequency ω and electric-field amplitude E_0:



$$\left[\lambda = \frac{2\pi c}{\omega} \right]$$

with $E_z(y,t) = E_0 \sin(\omega t - ky)$. The equation of motion for the electron is:

$$m\ddot{z} = -eE_0 \sin\omega t . \qquad \left[\ddot{z} \equiv \frac{d^2 z}{dt^2} \right]$$

mass of eletron ↑ ↑ electron charge

The instantaneous acceleration is:

$$a = \ddot{z} = -\frac{eE_0}{m} \sin\omega t .$$

The instantaneous power radiated by an accelerating charge (derived in 8.03 – hopefully) is given by

$$P = \frac{2}{3} \frac{e^2 a^2}{c^3} . \qquad \text{[cgs units]}$$

Evaluate this expression with the acceleration of the oscillating electron:

$$P = \frac{2}{3}\frac{e^4 E_0^2}{m^2 c^3} \sin^2 \omega t \quad .$$

As defined in lecture, the scattering cross section σ is given by

$$\sigma = \frac{\text{Scattered power}}{\text{Incident Flux}} \quad .$$

However, the incident energy flux in a plane electromagnetic wave is easily obtained from the Poynting vector (from 8.03):

$$\text{Flux} = \frac{c}{4\pi} E_0^2 \sin^2 \omega t$$

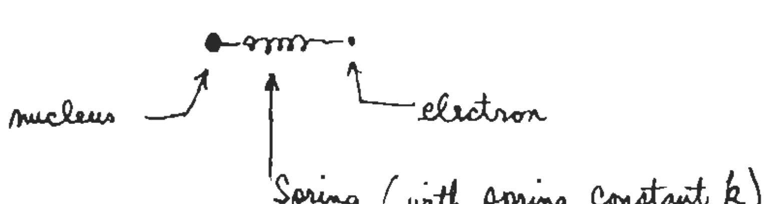
Thus, we find:

$$\boxed{\sigma_{\text{Thomson}} = \frac{8\pi}{3}\frac{e^4}{m^2 c^4} = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2}$$

$$\frac{e^2}{mc^2} \equiv \text{classical electron radius} \quad (2.8 \times 10^{-13}\ \text{cm})$$

Rayleigh Scattering

Consider a simple classical model of an atom



The equation of motion for the electron is the same as that given above for the free electron, with the addition of a "restoring force" term, $-kz$:

$$m\ddot{z} = -kz - eE_0 \sin\omega t$$

Try a solution of the form $z = A\sin\omega t$, where A is a constant to be determined.

$$-m\omega^2 A \sin\omega t = -kA\sin\omega t - eE_0 \sin\omega t,$$

which yields

$$A = \frac{(eE_0/m)\sin\omega t}{(\omega^2 - k/m)} .$$

The instantaneous acceleration ($\ddot{z} = -\omega^2 A \sin\omega t$) is:

$$a = \ddot{z} = -\frac{\omega^2 (eE_0/m)\sin\omega t}{(\omega^2 - k/m)} .$$

The cross section will be given by:

$$\sigma = \frac{\frac{2}{3}\frac{e^2 a^2}{c^3}}{\frac{c}{4\pi}E_0^2 \sin^2\omega t},$$

Scattered Power

Incident Flux

as in the previous case.

$$\sigma = \frac{8\pi}{3} \frac{e^4 \omega^4}{m^2 c^4 (\omega^2 - k/m)^2} \Rightarrow$$

$$\boxed{\sigma_{Rayleigh} = \sigma_{Thomson} \frac{\omega^4}{(\omega^2 - k/m)^2}}$$

In this expression, the quantity k/m is the square of the natural frequency of the electron-on-a-spring system, which we call ω_0. When the incident radiation has a frequency close to the natural frequency of the system, the cross section becomes very large! For the case where the incident radiation has a frequency well below that which is required to excite the atom, i.e., $\omega << \omega_0$, the cross section becomes:

$$\boxed{\sigma_{Rayleigh} \cong \sigma_{Thomson} \left(\frac{\omega}{\omega_0}\right)^4 = \sigma_{Thomson} \left(\frac{\lambda_0}{\lambda}\right)^4}$$