8.311: Electromagnetic Theory Problem Set # 10 Due: **4/23/04**

Retarded electromagnetic Green's function. Energy balance in radiation.

Reading: Schwinger, Chaps. 31, 32.

1. Charge moving at constant velocity. *(Schwinger, Problems 1 & 2, Chap 31)*

(a) A particle with charge e moves with constant velocity $\mathbf{v}$. Its position is given by $\mathbf{r}(t) = \mathbf{v}t$. Construct the potentials, ϕ and $\mathbf{A}$, in the Lorentz gauge, and show that

$$\phi = \frac{e}{\sqrt{(\mathbf{r} - \mathbf{v}t)^2 - \frac{v^2}{c^2}\mathbf{r}_\perp^2}}\,, \quad \mathbf{A} = \frac{\mathbf{v}}{c}\phi \tag{1}$$

where $\mathbf{r}_\perp$ is the component of particle radius vector perpendicular to the velocity.

(b) What are the electric and magnetic fields for this particle?

(Hint: A particle in uniform motion is not very different from a particle at rest. One can either follow the route suggested in the textbook and use Eqs. (31.20),(31.21), or apply a Lorentz transformation to the fields of a stationary particle.)

2. Potentials and fields of an arbitrarily moving charge. *(Schwinger, Problems 5 & 6, Chap 31)*

(a) The charge and current densities of a point charge are given by $\rho(\mathbf{r}, t) = e\delta(\mathbf{r} - \mathbf{r}(t))$, $\mathbf{j}(\mathbf{r}, t) = e\mathbf{v}(t)\delta(\mathbf{r} - \mathbf{r}(t))$. From the retarded potentials Eqs. (31.49),(31.50), derive the Lienard-Wiehert potentials

$$\phi(\mathbf{r}, t) = \frac{e}{|\mathbf{r} - \mathbf{r}(t')| - (\mathbf{r} - \mathbf{r}(t')) \cdot \frac{\mathbf{v}(t')}{c}}\,, \quad \mathbf{A}(\mathbf{r}, t) = \frac{\mathbf{v}(t')}{c}\phi(\mathbf{r}, t) \tag{2}$$

where the retarded time t' is defined by

$$t - t' = \frac{|\mathbf{r} - \mathbf{r}(t')|}{c} \tag{3}$$

(b) From potentials of part (a) compute $\mathbf{E}$ and $\mathbf{B}$. Express the answer as the sum of a "velocity" part (involving $\mathbf{v}$ only, and asymptotic to $1/r^2$), and an "acceleration" part (proportional to $\dot{\mathbf{v}}$ and asymptotic to $1/r$). Only the latter is significant for radiation.

3. Radiation of an electron moving in a magnetic field. *(Schwinger, Problem 1, Chap 32)*

A nonrelativistic particle of charge e and mass m moves in a uniform magnetic field $\mathbf{B}$. Suppose the motion is confined to the plane perpendicular to $\mathbf{B}$. Calculate the radiated power P and express it in terms of particle energy, $-P = dE/dt$. For the energy E, derive an ordinary differential equation

$$-\frac{dE}{dt} = \gamma E \tag{4}$$

and find γ in term of $\mathbf{B}$. For an electron, find $1/\gamma$ in seconds for a magnetic field of 10^4 gauss.

4. Radiation of a classical hydrogen atom. *(Schwinger, Problem 3, Chap 32)*

A nonrelativistic electron of charge e and mass m moves in a circular orbit under Coulomb forces produced by a proton. The average potential energy is related to the total energy by $E = \frac{1}{2}\bar{V}$.

Suppose that, as it radiates, the electron continues to move in a circle, and calculate the power radiated, and thereby $-dE/dt$, as a function of E (the relation is no longer linear). Integrate this result and find how long it takes for the energy to change from E_2 to E_1.

Show that the electron reaches the center in a finite time. Calculate how long it takes an electron to hit the proton if it starts from an initial radius of $r_0 = 10^{-8}$ cm. (This *radiation instability* of a classical atom was one of the reasons for the discovery of quantum mechanics.)

5. Dipole radiation.

(a) A nonrelativistic electron of charge e and mass m is driven by a time-dependent electric field $\mathbf{E} = E_0\hat{\mathbf{z}}\cos\omega t$. Show that the radiation is given by that of an oscillating dipole $\mathbf{d}(t) = e\mathbf{r}(t)$, and find the angular distribution of radiated power *averaged over oscillation cycle.*

(b) A nonrelativistic electron of charge e and mass m moves in a uniform and constant magnetic field $\mathbf{B}$. As in part a), show that the radiation is given by that of a rotating dipole $\mathbf{d}(t) = e\mathbf{r}(t)$, and find the time-averaged radiated power angular distribution.