Problem set #6

Prob 1 a) $\Phi(x,y) = \sum_{n\geq 1} \varphi_n(y) \sin k_n x \qquad k_n = \frac{\pi n}{a}$

$$\nabla^2\varphi = (\partial_x^2 + \partial_y^2)\varphi = \sum_n (\varphi_n'' - k_n^2 \varphi_n) \sin k_n x$$

$$\nabla^2\Phi = 0 \rightarrow \quad \varphi_n'' = k^2 \varphi_n \ , \ \varphi_n(y) = A_n e^{-k_n y}$$

B.C. $\sum_n A_n \sin k_n x = V \rightarrow A_n = \frac{2}{a}\int_0^a \sin k_n x \, V dx$

$$A_n = \frac{2V}{a k_n}(1 - \cos k_n a) = \begin{cases} 0 & n \text{ even} \\ 4V/a k_n & n \text{ odd} \end{cases}$$

$$\Phi(x,y) = \sum_{n\ odd} \frac{4V}{a k_n} e^{-k_n y} \sin k_n x = \mathrm{Im} \sum_{n\ odd} \frac{4V}{a k_n} e^{-k_n(y-ix)} = \mathrm{Im} \frac{2V}{\pi} \sum_{n\ odd} \frac{2}{n} W^n$$

$$\Phi(x,y) = \mathrm{Im}\, \frac{2V}{\pi} \ln \frac{1+W}{1-W} \qquad W = e^{-\pi(y-ix)/a}$$

$$\Phi(x,y) = \frac{2V}{\pi} \arg\left(\coth \frac{\pi z}{2a}\right) \qquad z = y - ix$$

b) $\Phi(x,y) = \sum_n A_n e^{-k_n y} \sin k_n x$, as in a)

$$\sigma(x)_{y=0} = -\frac{1}{4\pi}\partial_y \Phi\Big|_{y=0} = \frac{1}{4\pi}\sum_n A_n k_n \sin k_n x$$

B.C. $\sigma(x) = \sigma$ (constant) $\rightarrow A_n = \frac{2}{k_n a}\int_0^a 4\pi\sigma \sin k_n x \, dx$

$$A_n = \begin{cases} 0, & n \text{ even} \\ \frac{16\pi\sigma}{a k_n^2}, & n \text{ odd} \end{cases}, \quad \Phi(x,y) = \sum_{n\ odd} \frac{16 a \sigma}{\pi n^2} e^{-k_n y} \sin k_n x$$

c) $\Phi(x,y,z) = \sum_{n,m} \varphi_{nm}(z) \sin k_m x \sin k_n y \ , \ \varphi_{nm}'' = (k_n^2 + k_m^2)\varphi_{nm}$

$$\varphi_{nm}(z) = V \sinh K_{nm} z / \sinh K_{nm} a \longleftarrow K_{nm} = (k_n^2 + k_m^2)^{1/2}$$

$\Phi(\text{center}) = V/6$, by symmetry

Prob 2 In conductor, $E, B \propto e^{ik z}$, $k = \frac{\omega}{c}\sqrt{\epsilon}$

$\vec{B}_{\parallel} = \sqrt{\epsilon}\, \hat{n} \times \vec{E}_{\parallel}$ $\qquad \epsilon = 1 + \frac{4\pi\sigma i}{\omega}$

Same is true outside, at the surface, by continuity of $E_{\parallel}$, $B_{\parallel}$

$$\vec{E}_{\parallel} = \zeta\, \vec{B}_{\parallel} \times \hat{n} \qquad \zeta = (\epsilon)^{-1/2} \underset{\omega \ll \sigma}{=} (1-i)\left(\omega/8\pi\sigma\right)^{1/2}$$

In the limit $\zeta = 0$, $\underline{E_{\parallel} = 0}$, and Faraday's law yields $\oint E\,dl = \frac{i\omega}{c} B_{\perp} = 0 \rightarrow \underline{B_{\perp} = 0}$

any loop @ surface

Prob 3

$$E_x(r) = E_x^{(0)} \cos k_x x \sin k_y y \sin k_z z \qquad k_x = \frac{\pi n}{a}$$
$$E_y(r) = E_y^{(0)} \sin k_x x \cos k_y y \sin k_z z \qquad k_y = \frac{\pi m}{b}$$
$$E_z(r) = E_z^{(0)} \sin k_x x \sin k_y y \cos k_z z \qquad k_z = \frac{\pi p}{c}$$

$E_x = 0$ for $y = 0, b$, $z = 0, c$, $E_z = 0$ for $x = 0, a$, $y = 0, b$

$E_y = 0$ for $x = 0, a$, $y = 0, b$

B.C. fulfilled: $\underline{E_{\parallel} = 0}$

$$0 = \nabla \cdot E = -\left(E_x^{(0)} k_x + E_y^{(0)} k_y + E_z^{(0)} k_z\right) \sin k_x x \sin k_y y \sin k_z z$$

$$\underline{k_x E_x^{(0)} + k_y E_y^{(0)} + k_z^{(0)} E_z^{(0)} = 0}$$

$$\omega_{nmp} = \frac{\pi}{c}\left(\frac{n^2}{a^2} + \frac{m^2}{b^2} + \frac{p^2}{c^2}\right)^{1/2}, \quad n+m+p > 1$$

$$\vec{B} = \frac{c}{i\omega} \vec{\nabla} \times \vec{E} \rightarrow B_x = \frac{c}{i\omega}\left(k_y E_z^{(0)} - k_z E_y^{(0)}\right) \sin k_x x \cos k_y y \cos k_z z$$

$$B_y = \frac{c}{i\omega}\left(k_z E_x^{(0)} - k_x E_z^{(0)}\right) \cos k_x x \sin k_y y \cos k_z z$$

$$B_z = \frac{c}{i\omega}\left(k_x E_y^{(0)} - k_y E_x^{(0)}\right) \cos k_x x \cos k_y y \sin k_z z$$

Prob 4 a) $E(r,t) = \mathrm{Re}\left(E(r)\, e^{-i\omega t}\right)$, same for B

$U_E = \frac{1}{16\pi}\int \overset{*}{E} E\, d^3r$, same for U_B

$$\frac{\omega^2}{c^2}\int E^* E\, d^3r = -\int E^* \nabla^2 E\, d^3r = \int E^* \nabla\times(\nabla\times E)\, d^3r$$

Indeed, $\nabla\times(\nabla\times E) = \nabla(\underset{0}{\underbrace{\nabla\cdot E}}) - \nabla^2 E = -\nabla^2 E$

Integrating by parts,

$$\frac{\omega^2}{c^2}\int E^* E\, d^3r = \int (\nabla\times E^*)(\nabla\times E)\, d^3r = \frac{\omega^2}{c^2}\int B^* B\, d^3r$$

Thus, $U_E = U_B$

b) $\frac{\omega_1^2}{c^2}\int E_2^* E_1\, d^3r = -\int E_2^* \nabla^2 E_1\, d^3r = \int (\nabla\times E_2^*)(\nabla\times E_1)\, d^3r$

same as in a)

comparing to

$$\frac{\omega_2^2}{c^2}\int E_2^* E_1\, d^3r = \int (\nabla\times E_2^*)(\nabla\times E_1)\, d^3r$$

obtain $\omega_1 = \omega_2$ or $\underline{\int E_1^* E_2 = 0}$

Prob 5 choose z axis in the propagation direction

a) $E, B \propto e^{ik_z z - i\omega t}$

TM waves: $B_z = 0$ x, y components of Maxwell eqs:

$$\partial_y E_z - ik_z E_y = \frac{i\omega}{c} B_x, \quad -\partial_x E_z + ik_z E_x = \frac{i\omega}{c} B_y$$

$$ik_z B_y = \frac{i\omega}{c} E_x, \quad ik_z B_x = -\frac{i\omega}{c} E_y$$

Express $E_{x,y}$, $B_{x,y}$ through E_z:

$$(E_x, E_y) = \frac{ik_z}{k^2}\nabla_\perp E_z, \quad (B_x, B_y) = -\frac{i\omega}{ck^2}\left(\nabla\times E_z \hat{z}\right)$$

$k^2 = \frac{\omega^2}{c^2} - k_z^2$ obtain $\nabla_\perp^2 E_z + k^2 E_z = 0$, b.c. $E_z = 0$

Prob 5 continued

TE wave: $E_z = 0$

$$(B_x, B_y) = \frac{ik_z}{k^2} \nabla_\perp B_z, \quad (E_x, E_y) = \frac{i\omega}{ck^2} \nabla \times B_z \hat{z}$$

$$\nabla_\perp^2 B_z + k^2 B_z = 0, \quad \text{B.c.} \quad \partial B_z / \partial n = 0$$

b) Wave guide with rectangular crossection

TM modes: $E_z = E_z^{(0)} \sin k_m x \, \sin k_n y$

$$k_m = \frac{\pi m}{a}, \quad k_n = \frac{\pi n}{b}, \quad m > 0, \; n > 0$$

$$\omega = c\left[(\pi m/a)^2 + (\pi n/b)^2 + k_z^2\right]^{1/2}$$

TE modes: $B_z = B_z^{(0)} \cos k_m x \, \cos k_n y$

$$k_m = \pi m/a, \quad k_n = \pi n/b, \quad m+n > 0$$

$$\omega = c\left[(\pi m/a)^2 + (\pi n/b)^2 + k_z^2\right]^{1/2}$$

Prob 6 a) TEM mode: $E_z, B_z = 0$

$$k_z \hat{z} \times B_\perp = -\frac{\omega}{c} E_\perp, \quad k_z \hat{z} \times E_\perp = \frac{\omega}{c} B_\perp \rightarrow \omega = c|k_z|$$

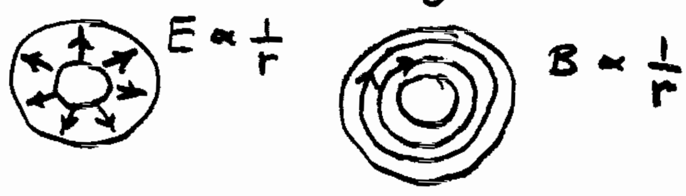
$$\rightarrow B_x = -E_y$$

$$\rightarrow B_y = E_x$$

$$\nabla_\perp^2 E_\perp = 0, \quad E = -\nabla \varphi$$

$\nabla^2 \varphi = 0$, b.c. $\varphi = \text{const}$ @ each conductor

Cylindrical geometry: $\varphi(r) = \varphi_0 \ln r/b$



b) At the end near load resistor R,

$\varphi_b - \varphi_a = RI$ $\qquad E(r) = \frac{E}{r}$, $B(r) = \frac{B}{r}$

$E = \frac{\varphi_b - \varphi_a}{\ln a/b}$ $\qquad B = \frac{2}{c} I$

reflection of a pulse from the R-end

$E = E_\rightarrow + E_\leftarrow$, $B = B_\rightarrow + B_\leftarrow$ $\qquad B_\rightarrow = E_\rightarrow$, $B_\leftarrow = -E_\leftarrow$

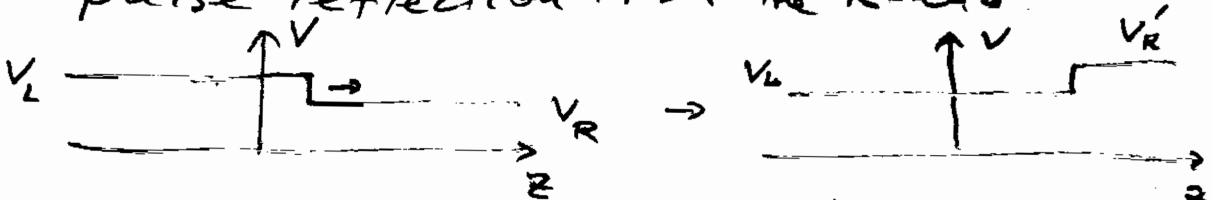
$$E_\rightarrow + E_\leftarrow = \frac{\varphi_{ba}}{\ln a/b} = \frac{RI}{\ln b/a} = \frac{cR}{2\ln b/a}\left(E_\rightarrow - E_\leftarrow\right)$$

define $u = 2\ln b/a \, R^{-1}$

$$\left(1+\frac{c}{u}\right)E_\leftarrow = \left(\frac{c}{u} - 1\right)E_\rightarrow \qquad E_\leftarrow = \frac{c-u}{c+u}E_\rightarrow$$

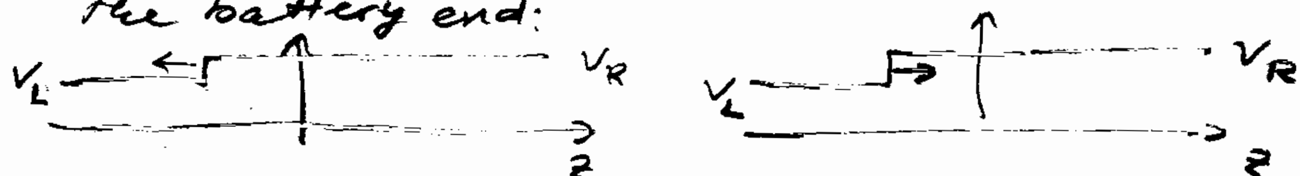
reflection coefficient $r = \frac{c-u}{c+u}$

pulse reflection from the R-end:



$V_R' - V_L = r(V_L - V_R)$

pulse reflection from the battery end:



$V_L' = V_2 = V_{battery}$

Voltage on R at $\frac{L}{c}(2n-1) < t < \frac{L}{c}(2n+1)$ $\quad n = 1, 2, 3, \ldots$

$$(V_n - V) = r(V - V_{n-1}) = (-r)^n(-V)$$

$$V_n = V - (-r)^n V$$

impedance matching