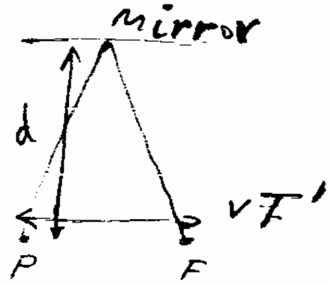
Problem Set #8

Problem 1

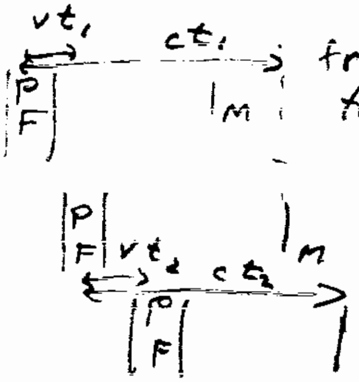
a)



$$T' = \frac{2d/c}{\sqrt{1-v^2/c^2}} = \beta T$$

$$(vT)^2 + (2d)^2 = (cT)^2$$

b)



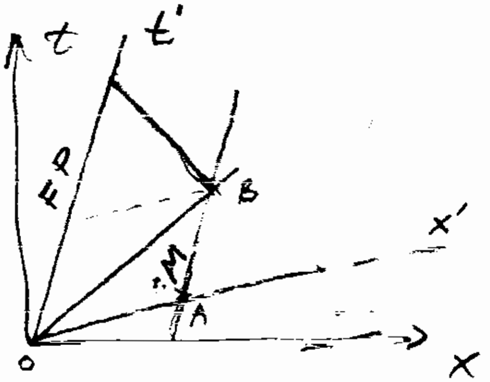
from F to M

$$t_1 = \frac{d'}{v+c} \qquad t_2 = \frac{d'}{c-v}$$

from M to P

$$T' = \frac{d'}{c+v} + \frac{d'}{c-v} = \frac{2d'/c}{1-v^2/c^2}$$

Consider world lines



From $c = \text{const}$, obtain $|AB| = |OA|$ which gives

$$\frac{T'}{T} = \frac{d}{d'}$$

combining with $\frac{T'}{T} = \frac{d'}{d}\beta^2$ (see above)

obtain $T' = \beta T$, $d' = \beta^{-1} d$

Problem 2 following Schwinger's derivation

$$x' = \beta(x + vt) \qquad x' + ct' = \sqrt{\frac{1+v/c}{1-v/c}}\,(x+ct)$$
$$t' = \beta\left(t + \frac{v}{c^2}x\right) \rightarrow \qquad x' - ct' = \sqrt{\frac{1-v/c}{1+v/c}}\,(x-ct)$$

For composition of two boosts,

$$\frac{1+u_{12}/c}{1-u_{12}/c} = \frac{1+v_1/c}{1-v_1/c}\cdot\frac{1+v_2/c}{1-v_2/c}$$ solving

$$u_{12} = \frac{v_1+v_2}{1+\frac{v_1 v_2}{c^2}}$$

Problem 3

Let us find motion with constant acceleration by considering charge moving along constant E field:

$\frac{dp}{dt}(=eE) = ma$ (use that $\vec{E}$ is invariant under L-boost w. $\vec{V} \parallel \vec{E}$)

acceleration in rocket frame

$p = ma(t-t_0)$

$\beta v = a(t-t_0) \rightarrow \frac{v^2}{a^2(t-t_0)^2} = 1 - \frac{v^2}{c^2}$

$$\left[\begin{array}{l} (x-x_0)^2 - c^2(t-t_0)^2 = \frac{c^4}{a^2} \leftarrow v = \frac{a(t-t_0)\,c}{(a^2(t-t_0)^2+c^2)^{1/2}} \\ \tau = \frac{c}{a}\,\mathrm{Asinh}\,\frac{a(t-t_0)}{c} \end{array}\right.$$

$\tau = 5\text{yrs}$, $t_{Earth} = \frac{c}{a}\sinh\frac{a\tau}{c}$, $\Delta x = \frac{c^2}{a}\left(\cosh\frac{a\tau}{c} - 1\right)$

a) $a\tau/c = 5.16$, $\frac{c}{a}\sinh\frac{a\tau}{c} = 85\,\text{yrs}$, rocket returns in 2316

b) $\Delta x = 1.5\cdot 10^{18}$ meters $= 158$ light yrs

Problem 4. a) For $|T| < d/c$ there is a ref. frame with no handicap, while for $|T| > d/c$ in any ref frame there is handicap

b) $v = \frac{T}{d} c^2$ in a) when $|T| < d/c$ with $\vec{v}$ along the y axis;

any frame is a solution for $|T| > d/c$.

Problem 5 a) $\frac{\delta L}{\delta p} = \dot{r} - v = 0 \rightarrow v = \dot{r}$

$$L = \frac{m}{2}\dot{r}^2 - e\phi + \frac{e}{c}\dot{r}\cdot A$$

Eqs. of motion: $\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r}$

$$\frac{d}{dt}\left(m\dot{r} - \frac{e}{c}A\right) = -e\nabla\phi + \frac{e}{c}\nabla(\dot{r}\cdot A)$$

$$\downarrow$$

$$\frac{dp_k}{dt} = eE + \frac{e}{c} v\times B \quad \text{(See Schwinger)}$$

with $p_k = m\dot{r}$

b) $H = pv - \frac{m v^2}{2} + e\phi - \frac{e}{c} v\cdot A$

$$\frac{\partial H}{\partial v} = p - mv - \frac{eA}{c} = 0 \quad \rightarrow \quad v = \frac{p - \frac{e}{c}A}{m}$$

$$\frac{dp}{dt} = -\nabla H = -\nabla\left(e\phi - \frac{e}{c}\dot{r}\cdot A\right) \Rightarrow \frac{dp_k}{dt} = eE + v\times B$$

$$p_k = m\dot{r}$$

$$\frac{dr}{dt} = \frac{1}{m}\left(p - \frac{e}{c}A\right)$$

c) see above

Problem 6 b) $\frac{\partial L}{\partial v} = \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}} - p + \frac{e}{c}A = 0$

$$H = pv + \beta mc^2 + e\phi - \frac{e}{c}v\cdot A = \left(\left(p - \frac{e}{c}A\right)^2 c^2 + (mc^2)^2\right)^{1/2} + e\phi$$

$$\frac{dp}{dt} = -\nabla H = -e\nabla\phi + \frac{e}{c}\nabla(v\cdot A) \qquad v = \frac{\partial E}{\partial p}$$

$$\downarrow$$

$$\frac{dr}{dt} = \nabla_p H = \frac{\partial E}{\partial p} \qquad \underline{\frac{dp_k}{dt} = eE + \frac{e}{c}v\times B}$$

$$p_k \equiv \beta m v$$

Problem 6 Choose $E \parallel \hat{x}$

$\frac{dp}{dt} = eE \rightarrow p_x(t) = p_{0x} + eEt, \quad p_{y,z} = \text{const}$

$E_k = c(p^2 + m^2c^2)^{1/2} = c\left((p_{0x} + eEt)^2 + p_{0y}^2 + p_{0z}^2 + m^2c^2\right)^{1/2}$

$\frac{d\vec{r}}{dt} = \frac{c\vec{p}}{(\quad)^{1/2}}$

$x(t) - x_0 = \frac{c}{eE}\left((p_{0x} + eEt)^2 + p_{0y}^2 + p_{0z}^2 + m^2c^2\right)^{1/2}$

$y(t) - y_0 = \frac{cp_{0y}}{eE} \operatorname{Asinh} \frac{p_{0x} + eEt}{(p_{0y}^2 + p_{0z}^2 + m^2c^2)^{1/2}}$

Same for $z(t)$

Initial condition $v = 0 \rightarrow p_{0x} = p_{0y} = p_{0z} = 0$

Trajectory:

$\frac{1}{c^2}(x - x_0)^2 - (t - t_0)^2 = \frac{m^2c^2}{(eE)^2}$

The motion is the same as in Problem 3 with constant acceleration $a = \frac{eE}{m}$ in particle frame