Suppl. notes, part II 8.311, 2004

Charge moving in a dielectric

$$S = \int dt\, L\,, \quad L = \int d^3r \left(\frac{E^2 \epsilon}{8\pi} - \frac{B^2}{8\pi}\right) + \frac{e}{c}\vec{A}(r(t))\cdot \vec{v}(t) - e\,\varphi(r(t))$$

Choose Coulomb gauge $\rightarrow$ $\vec{\nabla}\cdot\vec{A} = 0$

$$\int E^2 d^3r = \int \left(\frac{1}{c^2}\dot{A}^2 + (\nabla\varphi)^2\right) d^3r \rightarrow \text{eliminate } \varphi$$

Normal modes: $\vec{A}(r,t) = \sum_\lambda q_\lambda(t)\,\vec{A}_\lambda(r)$

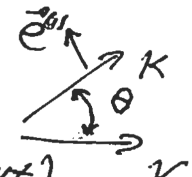
$$A_\lambda(r) = \sqrt{\frac{8\pi c^2}{\epsilon \underbrace{V}_{L^3}}}\,\vec{e}_\lambda \begin{cases}\cos k_\lambda r\\ \sin k_\lambda r\end{cases}$$

$$S = \sum_\lambda S_\lambda\,, \quad S_\lambda = \int dt \left(\frac{\dot{q}_\lambda^2}{2} - \frac{\omega_\lambda^2}{2} q_\lambda^2 + f_\lambda(t)\, q_\lambda\right)$$

$$\omega_\lambda = \frac{c}{\sqrt{\epsilon}}|k_\lambda|\,, \quad f_\lambda(t) = \frac{e}{c}\vec{A}_\lambda(r(t))\cdot v(t)$$

Consider motion with constant velocity v

$$r(t) = \begin{cases} vt\,, & t > 0\\ 0\,, & t < 0\end{cases}$$



polarizations:
$\vec{e}^{(1)}$ in the v-k plane
$\vec{e}^{(2)} \parallel \vec{v}\times\vec{k}$

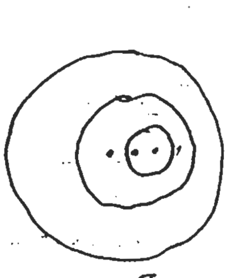
$$f_\lambda(t) = ev\sqrt{\frac{8\pi}{\epsilon V}}\,\sin\theta \begin{cases}\cos(|k_\lambda|\cos\theta\, vt)\\ \sin(|k_\lambda|\cos\theta\, vt)\end{cases}$$

$f_\lambda = 0$ for $\vec{e}^{(2)}$

driving frequency $\omega_f = |K|\cos\theta\, v$

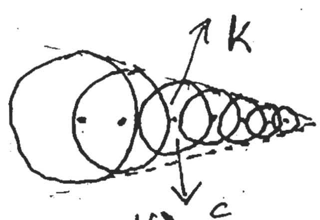
resonance $\omega_f = \omega_\lambda \rightarrow |k|\cos\theta\, v = \frac{|K|c}{\sqrt{\epsilon}}$

$$\boxed{\cos\theta = \frac{c}{v\sqrt{\epsilon}}}$$ can be satisfied for $v > \frac{c}{\sqrt{\epsilon}}$



$v < \frac{c}{\sqrt{\epsilon}}$

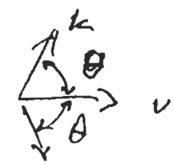
no radiation



$v > \frac{c}{\sqrt{\epsilon}}$

Charge radiating

Angular distribution @ cone



Radiated power (calculate for cos-modes then double the answer to account for sin-modes)

$$\ddot{q}_\lambda + \omega_\lambda^2 q_\lambda = \begin{cases} f_\lambda \cos\omega_f t, & t>0 \\ 0 & t<0 \end{cases}$$

$$q_\lambda(t) = \frac{f_\lambda}{\omega_\lambda^2 - \omega_f^2}\left(\cos\omega_f t - \cos\omega_\lambda t\right)$$

$$E_\lambda^{rad}(t) = \int_0^t f_\lambda \cos\omega_f t \; \dot{q}_\lambda(t) = \ldots = \frac{f_\lambda^2\,\omega_\lambda}{(\omega_\lambda^2-\omega_f^2)2} \; \frac{1-\cos(\omega_f-\omega_\lambda)t}{\omega_\lambda-\omega_f}$$

keep only resonance terms

$$E(t)_{total} = \sum_\lambda E_\lambda^{rad} = \sum_\lambda \frac{f_\lambda^2\,\omega_\lambda}{2(\omega_f+\omega_\lambda)} \; \frac{1-\cos(\omega_\lambda-\omega_f)t}{(\omega_\lambda-\omega_f)^2}$$

replace by $\pi t\,\delta(\omega_\lambda-\omega_f)$

$$P = \frac{dE_{total}}{dt} = \sum_\lambda \frac{\pi f_\lambda^2}{4}\,\delta(\omega_\lambda-\omega_f)$$

$$\sum_\lambda \ldots = V\int \frac{k^2\,dk}{(2\pi)^3}\,d\Omega \times 2 \times (1+0)\ldots$$

$d\Omega$: $\sin\theta\, d\theta\, d\varphi$, $0<\theta<\frac{\pi}{2}$ (since $\vec{k} \equiv -\vec{k}$), $0<\varphi<2\pi$

2: cos + sin

$(1+0)$: $\vec{e}_\lambda$

$$P = \int_0^\infty \frac{k^2\,dk}{(2\pi)^2}\int_0^{\pi/2} d(\cos\theta)\,(2\pi)\,\frac{8\pi e^2 v^2}{4\epsilon}\,\delta\!\left(|k|v\cos\theta - \frac{|k|c}{\sqrt{\epsilon}}\right)\sin^2\theta$$

$\sqrt{\epsilon}$: function of ω (and thus of k)

$$= \int_0^\infty k^2 dk \; \frac{e^2 v^2}{\epsilon |k| v}\left(1-\frac{c^2}{v^2\epsilon}\right) =$$

$$= \frac{e^2 v}{\epsilon}\int_0^\infty k\,\sin^2\theta_k\, dk$$

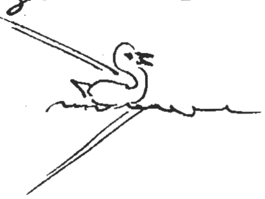
Spectral density of radiation $\quad \dfrac{dP}{d\omega} = \dfrac{e^2 v}{c^2}\sin^2\theta_\omega\;\omega$

$$\sin^2\theta_\omega = 1 - \frac{c^2}{v^2\epsilon_\omega}$$

1) $v<c$ → Čerenkov radiation (Tamm, Frank)

2) Superluminal charge in vacuum radiates (Sommerfeld)

3) generalizations



V-trace

Radiation at collision

consider only $\omega \ll \frac{2\pi}{\tau}$ ← collision time

For such ω results are universal, i.e. depend on $\vec{v}_1$ & $\vec{v}_2$, but not on the details of collision

$$\vec{v}(t) = \begin{cases} \vec{v}_1, & t<0 \\ \vec{v}_2, & t>0 \end{cases}$$

$$\vec{r}(t) = \begin{cases} \vec{v}_1 t, & t<0 \\ \vec{v}_2 t, & t>0 \end{cases}$$

$$\rightarrow f_\lambda(t) = \frac{\sqrt{8\pi}e}{\sqrt{V}} \cdot \vec{v}(t)\cdot\vec{e}_\lambda \begin{cases} \cos \vec{k}_\lambda\cdot\vec{r}(t) \\ \sin \vec{k}_\lambda\cdot\vec{r}(t) \end{cases}$$

V ← L^3 (volume)

$$\ddot{q} + \omega^2 q = f(t)$$

1) cos. modes: $t<0$ $\quad q(t) = \frac{f_1}{\omega^2-\omega_1^2}\cos\omega_1 t$

$\omega_{1,2} = \vec{k}\cdot\vec{v}_{1,2}$

$\omega = |k|c$

$t>0$ $\quad q(t) = \underbrace{\frac{f_2}{\omega^2-\omega_2^2}\cos\omega_2 t}_{\text{field of moving charge}} + \underbrace{q_c \cos\omega t}_{\text{radiated field}}$

find q_c from $q(t)$ & $\dot{q}(t)$ continuity at $t=0$

$$q_c = \frac{f_1}{\omega^2-\omega_1^2} - \frac{f_2}{\omega^2-\omega_2^2}$$

2) sin. modes: $t<0$ $\quad q(t) = \frac{f_1}{\omega^2-\omega_1^2}\sin\omega_1 t$

$t>0$ $\quad q(t) = \frac{f_2}{\omega^2-\omega_2^2}\sin\omega_2 t + q_s \sin\omega t$

$$q_s = \frac{\omega_1 f_1}{\omega(\omega^2-\omega_1^2)} - \frac{\omega_2 f_2}{\omega(\omega^2-\omega_2^2)}$$

Radiated energy $\Delta E^{rad}_{cos} = \frac{\omega^2}{2} q_c^2$ (cross-terms with frequencies $\omega_2 \pm \omega$ vanish under averaging over time: $\omega_2 \neq \omega$)

$\Delta E^{rad}_{sin} = \frac{\omega^2}{2} q_s^2$

$$\Delta E^{rad}_{c+s} = \frac{\omega^2}{2}(q_c^2+q_s^2) = \frac{4\pi e^2}{V\omega^2}\Bigg[\left(\frac{\vec{v}_1\cdot\vec{e}_\lambda}{1-(\vec{v}_1\cdot\vec{n})^2} - \frac{\vec{v}_2\cdot\vec{e}_\lambda}{1-(\vec{v}_2\cdot\vec{n})^2}\right)^2 + \left(\frac{(v_1\cdot n)(\vec{v}_1\cdot\vec{e}_\lambda)}{1-(v_1\cdot n)^2} - \frac{e(v_2\cdot n)(v_2\cdot e_\lambda)}{1-(v_2\cdot n)^2}\right)^2\Bigg]$$

$\vec{n} = \vec{k}/|k|$

$\frac{\omega_{1,2}}{\omega} = \vec{v}_{1,2}\cdot\vec{n}$

(suppress c)

Identity: $\sum_{i=1,2} (\vec{a}\cdot\vec{e}_i)^2 = \vec{a}^2 - (\vec{a}\cdot\vec{n})^2 = (\vec{a}\times\vec{n})^2$

$$\Delta E^{red} = V\int \frac{d^3k}{(2\pi)^3}\frac{1}{2}\frac{4\pi e^2}{V\omega^2}\left[\left(\frac{\vec{v}_1\times\vec{n}}{1-(v_1\cdot n)^2} - \frac{\vec{v}_2\times\vec{n}}{1-(v_2\cdot n)^2}\right)^2 + \right.$$
$(\vec{k} \equiv -\vec{k})$
$$\left. + \left(\frac{(v_1\cdot n)\,\vec{v}_1\times\vec{n}}{1-(v_1\cdot n)^2} - \frac{(v_2\cdot n)\,\vec{v}_2\times\vec{n}}{1-(v_2\cdot n)^2}\right)^3\right]$$

Write: $\frac{1}{1-(v_{1,2}\cdot n)^2} = \frac{1}{2}\left(\frac{1}{1-(v_{1,2}\cdot n)} + \frac{1}{1+(v_{1,2}\cdot n)}\right)$

$$\frac{v_{1,2}\cdot n}{1-(v_{1,2}\cdot n)^2} = \frac{1}{2}\left(\frac{1}{1-v_{1,2}\cdot n} - \frac{1}{1+v_{1,2}\cdot n}\right)$$

$$[\ldots] = \frac{1}{4}\left(\frac{\vec{v}_1\times\vec{n}}{1-v_1\cdot n} - \frac{\vec{v}_2\times\vec{n}}{1-v_2\cdot n} - \left\{\begin{matrix}\text{same,}\\ \vec{n}\to-\vec{n}\end{matrix}\right\}\right)^2 + \left(\frac{\vec{v}_1\times\vec{n}}{1-v_1\cdot n} - \frac{\vec{v}_2\times\vec{n}}{1-v_2\cdot n} + \left\{\vec{n}\to-\vec{n}\right\}\right)^2$$

$$= \frac{1}{2}\left(\frac{\vec{v}_1\times\vec{n}}{1-v_1\cdot n} - \frac{\vec{v}_1\times\vec{n}}{1-v_1\cdot n}\right)^2 + \frac{1}{2}\left(\frac{\vec{v}_1\times\vec{n}}{1+v_1\cdot n} - \frac{\vec{v}_2\times\vec{n}}{1+\vec{v}_2\cdot\vec{n}}\right)^2$$

the sum of $\vec{k}$ and $-\vec{k}$ contributions

$$\Delta E^{rad} = \int \frac{d^3k}{(2\pi)^3}\frac{4\pi e^2}{\omega^2}\frac{1}{2}\left(\frac{v_1\times n}{1-v_1\cdot n} - \frac{v_2\times n}{1-v_2\cdot n}\right)^2$$

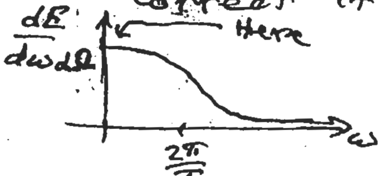
supress $\vec{k} \equiv -\vec{k} \to$ no $\frac{1}{2}$

restore c

$$\Delta E^{rad} = \int_0^\infty \frac{e^2 d\omega}{4\pi^2 c^3}\int d\Omega \left(\frac{\vec{v}_1\times\vec{n}}{1-\frac{1}{c}\vec{v}_1\cdot\vec{n}} - \frac{\vec{v}_2\times\vec{n}}{1-\frac{1}{c}\vec{v}_2\cdot\vec{n}}\right)^2$$

$$\frac{dE^{rad}}{d\omega\, d\Omega} = \frac{e^2}{4\pi^2 c^3}\left(\frac{\vec{v}_1\times\vec{n}}{1-\vec{v}_1\cdot\vec{n}/c} - \frac{\vec{v}_2\times\vec{n}}{1-v_2\cdot n/c}\right)^2$$

correct if $\frac{2\pi}{\omega} \gg \tau$, collision time



non.relativistic collision:

$$\frac{dE^{rad}}{d\omega\, d\Omega} = \frac{e^2}{4\pi^2 c^3}\left((\vec{v}_1-\vec{v}_2)\times\vec{n}\right)^2$$

Radiation Spectrum

- continuous, no characteristic frequency
- cut off at $\omega > \frac{2\pi}{\tau}$
- angular distribution: 1) $v_{1,2} \ll c$ → like dipole radiation
  2) $v_{1,2} \to c$ → very anisotropic, focused along $\vec{v}_{1,2}$

Compare: Infrared catastrophy in Quantum Electrodynamics

$\#\,\text{quanta} = \int^\infty \frac{d\omega}{\hbar\omega} I(\omega) \sim \ln\frac{1}{\tau v}$, diverges at $v \to 0$