8.321 Quantum Theory-I Fall 2017

Prob Set 7

1. For a particle of charge e and mass m moving in two dimensions in a uniform perpendicular magnetic field $\vec{B} = B\hat{z}$, we obtained the Landau level spectrum in two different ways in class. Here you will obtain it in a third way. Represent the magnetic field in terms of a vector potential in the "symmetric" gauge:

$$A_x = -\frac{By}{2}, \quad A_y = \frac{Bx}{2} \tag{1}$$

   (a) Check by taking the curl that this vector potential indeed corresponds to the magnetic field $\vec{B} = B\hat{z}$.

   (b) Solve for the distinct energy eigenvalues by relating the Hamiltonian to that of a $2d$ simple harmonic oscillator.

   (c) In the lowest Landau level, obtain the form of the energy eigenfunctions. You will find it convenient to work with the complex coordinates $z = x + iy$.

   (d) **Optional: Not for credit**

   For a two dimensional disc of radius R, argue how to recover the known answer for the degeneracy of the ground state.

2. Consider the Landau level problem for two dimensional particles with a dispersion $\epsilon(p) = \frac{p^2}{2m} + \lambda p^4$ with $\lambda > 0$. The detailed energy level structure will be different from that with the usual quadratic dispersion, and you need not determine it. This question is instead about the Landau level degeneracy. Show that the degeneracy of each allowed energy level is given by the same answer as at $\lambda = 0$.

3. A particle of charge e and mass m moves in a uniform magnetic field $\vec{B} = B\hat{z}$ and a uniform perpendicular electric field $\vec{E} = \mathcal{E}\hat{x}$.

   (a) Write the Hamiltonian for this system. For convenience in the calculations below use the Landau gauge $A_x = 0$, $A_y = Bx$, $A_z = 0$.

   (b) Show that the canonical momenta p_z, p_y are conserved.

   (c) If $\mathcal{E} = 0$ we get Landau levels. When $\mathcal{E} \neq 0$ find the exact eigenenergies. What happens to the Landau levels?

   (d) For each eigenstate with energy E, evaluate the velocities $v_z = \frac{\partial E}{\partial p_z}$ and $v_y = \frac{\partial E}{\partial p_y}$.

4. Consider two particles - one with charge $+q$ and the other with charge $-q$ - moving in a uniform magnetic field in two dimensions. The Hamiltonian is

$$H = \frac{\vec{\Pi}_1^2}{2m} + \frac{\vec{\Pi}_2^2}{2m} + V(\vec{x}_1 - \vec{x}_2) \tag{2}$$

Here $\vec{\Pi}_1 = \vec{p}_1 - q\vec{A}(\vec{x}_1)$, $\vec{\Pi}_2 = \vec{p}_2 + q\vec{A}(\vec{x}_2)$ are the kinematic momenta of the two particles. $\vec{A}$ is the vector potential corresponding to the uniform magnetic field $\vec{B} = B\hat{z}$, and $\vec{x}_{1,2}$ are the coordinates of the two particles. V is an attractive interaction between the two particles which you can take to have a harmonic form:

$$V(\vec{x}_1 - \vec{x}_2) = \frac{k}{2}|\vec{x}_1 - \vec{x}_2|^2 \tag{3}$$

Below you will consider this problem in center-of-mass $\vec{R} = \frac{\vec{x}_1 + \vec{x}_2}{2}$, and relative coordinates $\vec{x} = \vec{x}_1 - \vec{x}_2$.

   (a) Define the two momenta

$$\vec{Q} = \vec{\Pi}_1 + \vec{\Pi}_2 - q\vec{x} \times \vec{B} \tag{4}$$

$$\vec{p} = \frac{\vec{\Pi}_1 - \vec{\Pi}_2}{2} \tag{5}$$

   Show that the pairs $(\vec{R}, \vec{Q})$ and $(\vec{x}, \vec{p})$ are canonically conjugate. Check also that $[Q_i, p_j] = [Q_i, x_j] = [Q_i, Q_j] = [p_i, p_j] = [R_i, p_j] = 0$.

(b) Show that $\vec{Q}$ commutes with the Hamiltonian. Show that this has a classical analog by inspecting the classical equations of motion for the two particles (together with an identification of a classical quantity that corresponds to $\vec{Q}$).

(c) As $[\vec{Q}, H] = 0$ and the two components of $\vec{Q}$ commute with each other, the Hamiltonian may be diagonalized simultaneously with $\vec{Q}$. For a fixed value of $\vec{Q}$, show that the Hamiltonian may be rewritten as

$$H = \frac{\left(\vec{Q} + q\vec{x} \times \vec{B}\right)^2}{4m} + \frac{\vec{p}^2}{m} + V(\vec{x}) \tag{6}$$

(d) The energy spectrum will depend on $\vec{Q}$ and can be determined exactly. To keep things simple specialize to $\vec{Q} = 0$, and find the exact spectrum.

3

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