8.321 Quantum Theory-I Fall 2017

Prob Set 8

1. Three spins A, B, and C - each with magnitude 1/2 - are prepared in a state
$$|\psi\rangle = N\left(|\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\uparrow\uparrow\uparrow\rangle\right) \tag{1}$$
The first spin describes A, the second B, and the third C, all in the S_z basis.

   (a) Determine N so that the state is normalized.

   (b) Find the probability that a measurement of S_z for A gives $+\frac{\hbar}{2}$. Repeat for the same measurement for either B or for C.

   (c) Calculate the density matrix of A.

   (d) Use (c) to obtain the probabilities of the outcome of S_z measurements on A. Repeat for S_x measurements on A.

2. **Sakurai 3.10**

3. Consider two quantum spins in a state
$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0,+\rangle + |1,0\rangle\right) \tag{2}$$
Here $0, 1$ refer to spin eigenstates in the z basis and $\pm$ to spin eigenstates in the x basis. Find the Schmidt decomposition for this state, and the density matrix for each of the two spins.

4. Consider a quantum system A in a mixed state described by a density matrix ρ_A. It is possible to regard this mixed state as arising due to entanglement of A with some other system B such that the composite system $A + B$ is in a pure state $|\psi_{A+B}\rangle$. The state $|\psi_{A+B}\rangle$ is known as a 'purification' of ρ_A in B and satisfies

$$Tr_B|\psi_{A+B}\rangle\langle\psi_{A+B}| = \rho_A \tag{3}$$

The purification is clearly not unique. Show that any two purifications of ρ_A on B are related by a unitary operator acting only on subsystem B. In other words two different purifications $|\psi_{A+B}\rangle$ and $|\psi'_{A+B}\rangle$ satisfy

$$|\psi'_{A+B}\rangle = U_B|\psi_{A+B}\rangle \tag{4}$$

5. **von Neumann entropy**

The von Neumann entropy of a system described by a density matrix ρ is defined as

$$S = -Tr\rho\ln\rho \tag{5}$$

Prove the following properties of the von Neumann entropy of a quantum system.

(a) The von Neumann entropy is invariant under unitary transformations of the density matrix ρ.

(b) If ρ has D non-vanishing eigenvalues then

$$S(\rho) \leq logD \tag{6}$$

with equality only when all the non-zero eigenvalues are equal. This means that the entropy is maximized when the quantum state is chosen randomly.

(c) If ρ_1 and ρ_2 describe two density matrices for a system, then first show that $\rho(\lambda) = \lambda\rho_1 + (1-\lambda)\rho_2$ is also a legitimate density matrix. Next show that

$$S(\rho(\lambda)) \geq \lambda S(\rho_1) + (1-\lambda)S(\rho_2) \tag{7}$$

This means that the more ignorant we are about how the state was prepared the higher the von Neumann entropy.

(d) For a composite system $A+B$ in a general mixed state described by density matrix ρ_{A+B},

$$S(\rho_{A+B}) \leq S(\rho_A) + S_(\rho_B) \tag{8}$$

with equality for uncorrelated systems where the full density matrix factorizes.