

# 8.321 Quantum Theory-I Fall 2016

Final Exam

Dec 19, 2016

## 1. (20 points)

- (a) Show that the density matrix for a spin-1/2 system can be written as

$$\rho = \frac{1}{2} (1 + \vec{n} \cdot \vec{\sigma}) \quad (1)$$

where  $\vec{n}$  is a real three component vector. Is there any restriction on  $\vec{n}$ ?

- (b) What condition does  $\vec{n}$  satisfy for a pure state?  
(c) What condition does  $\vec{n}$  satisfy for a maximally random state?

## 2. 20 points

A particle of charge  $e$  and mass  $m$  moves in a uniform magnetic field  $\vec{B} = B\hat{z}$  and a uniform perpendicular electric field  $\vec{E} = \mathcal{E}\hat{x}$ .

- (a) Write the Hamiltonian for this system. For convenience in the calculations below use the Landau gauge  $A_x = 0, A_y = Bx, A_z = 0$ .  
(b) Show that the canonical momenta  $p_z, p_y$  are conserved.  
(c) If  $\mathcal{E} = 0$  we get Landau levels. When  $\mathcal{E} \neq 0$  find the exact eigenenergies. What happens to the Landau levels?  
(d) For each eigenstate with energy  $E$ , evaluate the velocities  $v_z = \frac{\partial E}{\partial p_z}$  and  $v_y = \frac{\partial E}{\partial p_y}$ .

3. (30 points)

It is well known that Sodium vapor emits yellow light at a wavelength of approximately  $590nm$ . Close study reveals that the emission actually occurs in two very close wavelengths. The atomic structure of Na is such that the  $1s$ ,  $2s$ , and  $2p$  orbitals are fully filled and there is a single electron in the  $n = 3$  orbitals. Unlike a Hydrogen atom, the  $3s$  orbital has lower energy than the  $3p$  orbitals. The observed yellow light is due to a  $3p$  to  $3s$  transition from an excited Na atom (where the outer electron occupies a  $3p$  orbital) to the ground state (one where the outer electron occupies the  $3s$  orbital).

In this problem you will explain the splitting of the Na emission line by focusing on the energy levels of the outer electron. Crucial to the story is the presence of spin-orbit coupling in the Hamiltonian of the form

$$H_{SO} = \frac{\kappa}{\hbar^2} \vec{L} \cdot \vec{S} \quad (2)$$

Here  $\vec{L}$  is the orbital angular momentum and  $\vec{S}$  is the electron spin.  $\kappa$  is a *positive* constant. Assume that in the absence of this coupling the  $3s$ ,  $3p$  levels have energy  $E_{3s}^{(0)}$ ,  $E_{3p}^{(0)}$  respectively with  $E_{3p}^{(0)} > E_{3s}^{(0)}$ . You may assume that  $\kappa \ll E_{3p}^{(0)} - E_{3s}^{(0)}$ .

- (a) Calculate the energy levels of the  $3s$  and  $3p$  orbitals when  $H_{SO}$  is included. Indicate the correct degeneracies for each level.
- (b) Obtain an expression for the splitting of the emission line in terms of  $\kappa$ . It is known from experiments that the splitting is about  $0.6nm$ . Use this to estimate the strength of the spin-orbit coupling  $\kappa$  in units of  $eV$ .
- (c) If now a weak magnetic field  $\vec{B} = B\hat{z}$  is turned on, it will couple to both the orbital motion and the spin of the electron through the term

$$H_B = \frac{eB}{2m} (L_z + gS_z) \quad (3)$$

Here  $g = 2$  (to a very good approximation). To leading order in  $B$  indicate how the energy levels found above will further split. You do not need to do an explicit calculation.

- (d) With  $B \neq 0$  how many distinct wavelengths will the emission spectrum consist of? (Note: for an allowed transition the quantum number  $J_z = L_z + S_z$  can only change by 0 or by  $\pm 1$  .)

4. (30 points)

In **two** space dimensions the angular momentum  $L$  is a single number (rather than a vector). We may define it to be the generator of two dimensional rotations. If the angular coordinate is denoted by  $\theta$ , then  $L$  generates shifts of  $\theta$ .

- (a) Explain why we can take

$$L = -i\hbar \frac{\partial}{\partial \theta} \quad (4)$$

- (b) Given Eqn. 4 determine the eigenvalues and corresponding eigenfunctions of  $L$ .

- (c) A rigid rotator in two space dimensions is described by the Hamiltonian

$$H_0 = \frac{L^2}{2I} \quad (5)$$

where  $I$  is the moment of inertia. Find the eigenvalues and their degeneracies.

- (d) The rotator is perturbed by a potential of the form

$$V = V_0 \cos(2\theta) \quad (6)$$

Find the shift of the ground state energy to leading non-vanishing order in  $V_0$ . Repeat for the first excited state.

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