Lecture 13 (Oct. 25, 2017)

13.1 Aharonov–Bohm Effect

Recall from last lecture the setup for the Aharonov–Bohm effect: we have a cylindrical shell through which runs a solenoid, producing a flux

$$\Phi = \int_{\text{core}} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} \tag{13.1}$$

through the core of the shell. There is no magnetic field inside the cylindrical shell, only through its core. As we saw, even though the magnetic field within the shell vanishes, it is impossible to choose the vector potential $\boldsymbol{A}$ to vanish inside the cylindrical shell. To see why, we took a closed contour C within the shell that enclosed the core, and noted that

$$\oint_C \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{\ell} = \int_\Sigma (\nabla \times \boldsymbol{A}) \cdot \mathrm{d}\boldsymbol{S} = \int_\Sigma \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = \Phi\,, \tag{13.2}$$

where Σ is the area bounded by the contour C, i.e., $C = \partial\Sigma$. This shows us that $\boldsymbol{A}$ cannot vanish everywhere within the shell.

In particular, let us consider the contour C to be a circular loop of radius R. The line integral above has contributions only from the azimuthal component of the vector potential. Thus, we should be able to produce the magnetic field coming from the solenoid by taking a vector potential of the form

$$\boldsymbol{A} = A_\theta \hat{\boldsymbol{e}}_\theta\,. \tag{13.3}$$

By cylindrical symmetry, it is reasonable to take A_θ to be independent of θ. We then have

$$\oint_C \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{\ell} = A_\theta(2\pi R) = \Phi\,, \tag{13.4}$$

which yields

$$A_\theta = \frac{\Phi}{2\pi R}\,. \tag{13.5}$$

Thus, a vector potential that produces the desired magnetic field is

$$\boldsymbol{A}(r) = \frac{\phi}{2\pi r}\hat{\boldsymbol{e}}_\theta\,, \tag{13.6}$$

where r is the radial cylindrical coordinate. Note that there are infinitely many vector potentials that will produce the same magnetic field, all related to one another by gauge transformations.

Because we have a nonzero vector potential, there will be nontrivial quantum mechanical behavior. Thinking in terms of the propagator

$$K(\boldsymbol{x}_f, t_f; \boldsymbol{x}_i, t_i) = \int [\mathcal{D}\boldsymbol{x}]\; e^{iS[\boldsymbol{x}(t)]/\hbar}\,. \tag{13.7}$$

The action is

$$\begin{aligned} S[\boldsymbol{x}(t)] &= \int \mathrm{d}t\; L(\boldsymbol{x}, \dot{\boldsymbol{x}}, t) \\ &= \int \mathrm{d}t\, \left(\frac{1}{2}m\dot{\boldsymbol{x}}^2 + \frac{e}{c}\boldsymbol{A} \cdot \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right) \\ &= \int \mathrm{d}t\, \frac{1}{2}m\dot{\boldsymbol{x}}^2 + \frac{e}{c}\int \mathrm{d}\boldsymbol{x} \cdot \boldsymbol{A}(\boldsymbol{x}) \\ &= S_0 + \frac{e}{c}\int \mathrm{d}\boldsymbol{x} \cdot \boldsymbol{A}(\boldsymbol{x})\,, \end{aligned} \tag{13.8}$$

where S_0 is the action without the presence of any magnetic field. Thus, the amplitude to propagate from $(\boldsymbol{x}_i, t_i)$ to $(\boldsymbol{x}_f, t_f)$ is equal to the amplitude with no magnetic field multiplied by the phase

$$\exp\left(\frac{ie}{\hbar c}\int_{\boldsymbol{x}_i}^{\boldsymbol{x}_f} \mathrm{d}\boldsymbol{x}\cdot\boldsymbol{A}\right). \tag{13.9}$$

So far this statement is general; let us now specialize to the case of the Aharonov–Bohm setup. Consider two points A and B distributed symmetrically around the cylindrical shell, and both lying in the same plane orthogonal to the axis of the cylindrical shell. We consider two types paths from A to B within this plane: those passing around the left of the cylindrical shell, and those passing around the right of the cylindrical shell. Note that our choice of a particular path around the left of the shell does not affect the phase accumulated along this path. To see this, note that we could choose two such paths and combine them to form a closed loop which encloses no magnetic flux; the closed line integral of $\boldsymbol{A}$ along this loop therefore vanishes, showing that the phase picked up on the two paths must be equal. In other words, for any two paths P_{L} and P'_{L} passing around the left, we have

$$\int_{P_{\mathrm{L}}} \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{x} = \int_{P'_{\mathrm{L}}} \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{x}\,. \tag{13.10}$$

This is similarly true for the paths around the right of the shell.

The amplitudes when traveling along the left and right paths P_{L} and P_{R} are therefore

$$\begin{aligned} a(P_{\mathrm{L}}) &= a_0 e^{\frac{ie}{\hbar c}\int_{P_{\mathrm{L}}}\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{x}}\,, \\ a(P_{\mathrm{R}}) &= a_0 e^{\frac{ie}{\hbar c}\int_{P_{\mathrm{R}}}\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{x}}\,. \end{aligned} \tag{13.11}$$

If we consider the difference of the additional phase factors, we have

$$\int_{P_{\mathrm{R}}} \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{x} - \int_{P_{\mathrm{L}}} \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{x} = \oint \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{x} = \Phi \neq 0\,. \tag{13.12}$$

The total propagator is thus

$$\begin{aligned} K &= K_{\mathrm{L}} e^{\frac{ie}{\hbar c}\int_{P_{\mathrm{L}}}\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{x}} + K_{\mathrm{R}} e^{\frac{ie}{\hbar c}\int_{P_{\mathrm{R}}}\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{x}} \\ &= e^{\frac{ie}{\hbar c}\int_{P_{\mathrm{L}}}\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{x}}\Big(K_{\mathrm{L}} + K_{\mathrm{R}} e^{ie\Phi/\hbar c}\Big)\,. \end{aligned} \tag{13.13}$$

The probability of propagating from point A to point B is then

$$\mathrm{Prob}(A\to B) = |K_{\mathrm{L}}|^2 + |K_{\mathrm{R}}|^2 + 2K_{\mathrm{L}}K_{\mathrm{R}}\cos\left(\frac{e\Phi}{\hbar c}\right). \tag{13.14}$$

This probability oscillates as a function of the flux Φ, even though classical trajectories do not see the magnetic field. We see that this oscillation depends only on the flux Φ, and not on our choice of gauge, as is required by gauge invariance.

We have derived this effect using the path integral approach. We could have also reached this result by using the Schödinger equation. The energy levels and wavefunctions for a particle in a ring threaded by flux Φ are dependent on Φ.

This result leads us to the following question: the vector potential is a (redundant) mathematical construct that we use to describe the physics, but the actually physical quantities are the electromagnetic fields. How can it be the case that the vector potential makes an imprint on the physics, when we know that the vector potential by itself is not gauge invariant? To reconcile the

Aharonov–Bohm effect and the non-reality of the vector potential, we must realize that the physics of a quantum particle is sensitive not only to the local magnetic field, but also the global properties of the magnetic field. This is a departure from the classical intuition: quantum mechanics requires nonlocal effects. Formulating the theory in the way that we have, using the vector potential, allows us to define the theory only locally and find the same results. We have formulated these nonlocal effects in terms of a purely local description, but the price we pay is that there is no unique description of the formalism, because we have gauge redundancy.

The same approach can be seen in quantum field theory. With quantum electrodynamics, we describe the physics of charged particles and photons. We all know that two charged particles interact via the Coulomb interaction, which is nonlocal; we can think of the introduction of photons as taming this nonlocality in the same way as the introduction of the vector potential tamed the nonlocality in the Aharonov–Bohm setup.

13.2 Magnetic Monopoles

Suppose that magnetic monopoles exist. This requires a modification to Gauss's law for magnetism,

$$\nabla \cdot \boldsymbol{B} = 0 \to \nabla \cdot \boldsymbol{B} = 4\pi\rho_M \,, \tag{13.15}$$

where ρ_M is the density of magnetic charge (as well as modifications to Faraday's law and the Lorentz force law). A magnetic monopole is a single magnetic charge of strength g. If there is a magnetic monopole of magnetic charge g at the origin, then

$$\boldsymbol{B} = \frac{g}{r^2}\hat{\boldsymbol{e}}_r \,. \tag{13.16}$$

What is the fate of a quantum mechanical particle of electric charge e moving in the presence of a magnetic monopole? We immediately realize a problem. We typically describe the magnetic field in quantum mechanics via the vector potential, but the existence of the vector potential is predicated on the fact that the divergence of the magnetic field vanishes. In a world where magnetic monopoles exist, the magnetic field does not have a representation as the curl of a vector potential.

Let's think about this problem in the presence of a single magnetic monopole sitting at the origin. In this situation, $\nabla \cdot \boldsymbol{B} = 0$ everywhere except at the origin. The implication is that if we exclude the point at the origin, then we can *locally* find a vector potential $\boldsymbol{A}$ such that $\boldsymbol{B} = \nabla \times \boldsymbol{A}$. To see more clearly why we cannot describe the magnetic field everywhere with a vector potential, consider a sphere of radius r centered at the origin. Due to the presence of the magnetic monopole at the origin, we see that

$$\int_{\text{sphere}} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = 4\pi g \,. \tag{13.17}$$

If we were able to write $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ with $\boldsymbol{A}$ well-defined *everywhere*, then we would have

$$\int_{\text{sphere}} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = \int \nabla \cdot \boldsymbol{B} \,\mathrm{d}V = \int \nabla \cdot (\nabla \times \boldsymbol{A}) \,\mathrm{d}V = 0 \,, \tag{13.18}$$

which yields a contradiction.

Consider a circle C on the surface of the sphere at angle θ. The circle divides the sphere into two caps; the magnetic flux through the upper cap of the sphere bounded by this circle is

$$\Phi = 2\pi \int \frac{g}{r^2} r^2 \,\mathrm{d}(\cos\theta) = 2\pi g(1 - \cos\theta) \,. \tag{13.19}$$

If there exists a vector potential

$$\boldsymbol{A} = A_\phi \hat{\boldsymbol{e}}_\phi \,, \tag{13.20}$$

then

$$\oint_C \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{\ell} = 2\pi r \sin\theta A_\phi \,. \tag{13.21}$$

On the other hand,

$$\oint_C \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{\ell} = \int_\Sigma \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = 2\pi g(1 - \cos\theta) \,, \tag{13.22}$$

where Σ is the upper cap bounded by C, i.e., $C = \partial\Sigma$. Thus, we have

$$A_\phi = \frac{g(1-\cos\theta)}{r\sin\theta} \,. \tag{13.23}$$

Note that this ratio is well-behaved as $\theta \to 0$, because the numerator goes to zero more quickly than the denominator, but the ratio blows up as $\theta \to \pi$.