Recitation 2 (Sep. 15, 2017)

This discussion closely follows "A Modern Approach to Quantum Mechanics" by Townsend.

2.1 Dirac Delta Distribution

The Dirac delta "function" is the object $\delta(x-x_0)$ defined by the property that for all smooth (and compactly supported) functions $f(x)$,

$$\int_{-\infty}^{\infty} \mathrm{d}x\ f(x)\delta(x-x_0) = f(x_0)\,. \tag{2.1}$$

This is not, in fact, a function, but rather a *distribution*, which is an object that can be integrated against test functions.

Nevertheless, we as physicists will talk about it as though it is a function, and make statements about it outside the context of integration against a test function. The first such statement is

$$\delta(x-x_0) = 0\,, \quad x \neq x_0\,, \tag{2.2}$$

which we conclude by examining the defining property of the Dirac delta function. By setting $f(x) = 1$, we also have

$$\int_{-\infty}^{\infty} \mathrm{d}x\ \delta(x-x_0) = 1\,. \tag{2.3}$$

In order to discuss the Dirac delta function as though it were actually a proper function with these two properties, we can think of it as the limit of a sequence of functions with unit area and progressively sharper and sharper peaks around the point x_0. One such description of the Dirac delta function is

$$\delta(x) = \lim_{\lambda\to\infty} \frac{\sin(\lambda x)}{\pi x}\,. \tag{2.4}$$

The function $\sin(\lambda x)/(\pi x)$ has unit area for all values of λ, and has a width of order $1/\lambda$. Furthermore, it is well-behaved for all finite values of λ. Using

$$\frac{\sin(\lambda x)}{\pi x} = \frac{1}{2\pi}\int_{-\lambda}^{\lambda} \mathrm{d}k\ e^{ikx}\,, \tag{2.5}$$

we find another expression for the delta function,

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k\ e^{ikx}\,. \tag{2.6}$$

Thus, we see that we can define the Dirac delta function as the Fourier transform (or the inverse Fourier transform) of the identity operator.

We can now derive several useful properties of the delta function. First, using change of variables, we see that for any nonzero $a \in \mathbb{R}$,

$$\begin{aligned}
\int_{-\infty}^{\infty} \mathrm{d}x\ f(x)\delta(ax) &= \begin{cases} \frac{1}{a}\int_{-\infty}^{\infty} \mathrm{d}u\ f\big(\frac{u}{a}\big)\delta(u)\,, & a>0 \\ \frac{1}{a}\int_{\infty}^{-\infty} \mathrm{d}u\ f\big(\frac{u}{a}\big)\delta(u)\,, & a<0 \end{cases} \\
&= \frac{1}{|a|}\int_{-\infty}^{\infty} \mathrm{d}u\ f\Big(\frac{u}{a}\Big)\delta(u) \\
&= \frac{1}{|a|} f(0)\,.
\end{aligned} \tag{2.7}$$

Thus, we conclude that

$$\delta(ax) = \frac{1}{|a|}\delta(x)\,. \tag{2.8}$$

We can use this result to derive a more general identity. Suppose that $g(x)$ is a function that has a single zero at x_0, $g(x_0) = 0$. We can then expand $g(x)$ in a Taylor series about $x = x_0$ to yield

$$g(x) = g(x_0) + g'(x_0)(x - x_0) + \cdots = g'(x_0)(x - x_0) + \cdots\,, \tag{2.9}$$

where $g'(x) = \frac{\mathrm{d}g(x)}{\mathrm{d}x}$. Using Eq. (2.8), we see then that

$$\delta(g(x)) = \frac{1}{|g'(x_0)|}\delta(x - x_0)\,. \tag{2.10}$$

Note that we can ignore the higher0order terms in the series expansion because the delta function vanishes except at $x = x_0$. More generally, if $g(x)$ has zeroes at $x = x_1, \ldots, x_n$, then

$$\delta(g(x)) = \sum_{i=1}^{n} \frac{1}{|g'(x_i)|}\delta(x - x_i)\,. \tag{2.11}$$

A particular example of Eq. (2.11) that comes up frequently in physics is

$$\delta\big(x^2 - a^2\big) = \frac{1}{2|a|}[\delta(x - a) + \delta(x + a)]\,. \tag{2.12}$$

Finally, we can consider the integral and derivative of the delta function. The integral of the delta function satisfies

$$\int_{-\infty}^{x} \mathrm{d}y\; \delta(y - a) = \begin{cases} 0\,, & x < a \\ 1\,, & x > a \end{cases} := \theta(x - a)\,, \tag{2.13}$$

where $\theta(x - a)$ is the unit step function, called the *Heaviside step function.* The derivative of the Dirac delta function can only be defined in the context of integration against a compactly supported, smooth function. In this case, we evaluate the integral against $\delta'(x)$ using integration by parts:

$$\int_{-\infty}^{\infty} \mathrm{d}x\; f(x)\delta'(x) = f(x)\delta(x)\big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \mathrm{d}x\; f'(x)\delta(x) = -f'(0)\,, \tag{2.14}$$

where in the last step we have used the fact that $f(x)$ must vanish at $\pm\infty$ because it is compactly supported. This defines the action of integration against $\delta'(x)$.

2.2 Gaussian Integrals

As we continue to discuss systems with continuous spectra, we will often find ourselves needing to evaluate Gaussian integrals. We will now discuss the standard tricks used to evaluate such integrals.

First, we consider the integral

$$I(a) = \int_{-\infty}^{\infty} \mathrm{d}x\; e^{-ax^2}\,, \tag{2.15}$$

with $\mathrm{Re}\, a > 0$. We can evaluate this integral by squaring it and converting to polar coordinates:

$$\begin{aligned} I^2(a) &= \int_{-\infty}^{\infty} \mathrm{d}x\, e^{-ax^2} \int_{-\infty}^{\infty} \mathrm{d}y\, e^{-ay^2} \\ &= \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{d}y\, e^{-a\left(x^2+y^2\right)} \\ &= \int_0^{2\pi} \mathrm{d}\theta \int_0^{\infty} r\, \mathrm{d}r\, e^{-ar^2} \\ &= \frac{\pi}{a}\,. \end{aligned} \tag{2.16}$$

Thus,

$$I(a) = \sqrt{\frac{\pi}{a}}\,. \tag{2.17}$$

Next, we consider the more general integral

$$I(a,b) = \int_{-\infty}^{\infty} \mathrm{d}x\, e^{-ax^2+bx}\,. \tag{2.18}$$

Completing the square in the exponent, we can write this as

$$I(a,b) = \int_{-\infty}^{\infty} \mathrm{d}x\, e^{-a\left(x-\frac{b}{2a}\right)^2+\frac{b^2}{4a}}\,. \tag{2.19}$$

We can then shift the integral by $x \to x + \frac{b}{2a}$, yielding

$$I(a,b) = e^{\frac{b^2}{4a}} \int_{-\infty}^{\infty} \mathrm{d}x\, e^{-ax^2} = e^{\frac{b^2}{4a}} \sqrt{\frac{\pi}{a}}\,. \tag{2.20}$$

We can also evaluate integrals of the form

$$\int_{-\infty}^{\infty} \mathrm{d}x\, x^2 e^{-ax^2} = -\frac{\mathrm{d}}{\mathrm{d}a} \int_{-\infty}^{\infty} \mathrm{d}x\, e^{-ax^2} = -\frac{\mathrm{d}}{\mathrm{d}a} I(a) = \frac{1}{2}\sqrt{\frac{\pi}{a^3}} \tag{2.21}$$

and

$$\int_{-\infty}^{\infty} \mathrm{d}x\, x^4 e^{-ax^2} = \frac{\mathrm{d}^2}{\mathrm{d}a^2} \int_{-\infty}^{\infty} \mathrm{d}x\, e^{-ax^2} = \frac{\mathrm{d}^2}{\mathrm{d}a^2} I(a) = \frac{3}{4}\sqrt{\frac{\pi}{a^5}}\,. \tag{2.22}$$

This approach can be extended in the obvious way to evaluate integrals with higher even powers of x in the integrand. Note that integrals with odd powers of x in the integrand vanish by symmetry.

Finally, we can use contour integration to generalize the result in Eq. (2.17) to the case $\mathrm{Re}\, a \geq 0$. Consider the closed contour integral

$$\oint \mathrm{d}z\, e^{-az^2} \tag{2.23}$$

for positive $a \in \mathbb{R}$ over the contour shown in Fig. 1. The integrand is analytic within the contour, and so the residue theorem tells us that

$$\oint \mathrm{d}z\, e^{-az^2} = 0\,. \tag{2.24}$$

At any point on the contour, we can parametrize z as

$$z = re^{i\theta} = r(\cos\theta + i\sin\theta)\,. \tag{2.25}$$

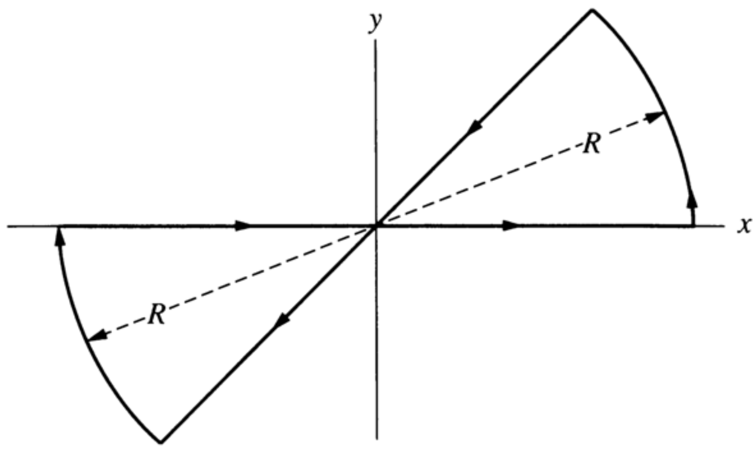

Figure 1: A closed contour in the complex z plane.

On the two circular arcs of the contour, the integrand is then

$$e^{-aR^2 e^{i2\theta}} = e^{-aR^2[\cos(2\theta)+i\sin(2\theta)]} . \tag{2.26}$$

As $R \to \infty$, this integrand goes exponentially to zero for $0 < \theta < \pi/4$ and $\pi < \theta < 5\pi/4$, which are the ranges of θ on the circular arcs of the contour. Thus, in this limit the circular arcs do not contribute to the integral.

In the $R \to \infty$ limit, we parametrize the diagonal line by $z = re^{i\pi/4}$ with r running from $+\infty$ to $-\infty$, and the horizontal line by $z = x$ with x running from $-\infty$ to $+\infty$. Making these changes of variables, we then have

$$\oint \mathrm{d}z\, e^{-az^2} = \int_{-\infty}^{\infty} \mathrm{d}x\, e^{-ax^2} + e^{i\pi/4} \int_{\infty}^{-\infty} \mathrm{d}r\, e^{-iar^2} = 0 \,, \tag{2.27}$$

where we have used $e^{i\pi/2} = i$. Thus,

$$\int_{-\infty}^{\infty} \mathrm{d}r\, e^{-iar^2} = e^{-i\pi/4} \int_{-\infty}^{\infty} \mathrm{d}x\, e^{-ax^2} = e^{-i\pi/4}\sqrt{\frac{\pi}{a}} = \sqrt{\frac{\pi}{ia}} \,, \tag{2.28}$$

so long as we take the branch where $\sqrt{i} = e^{i\pi/4}$. This matches Eq. (2.17) with $a \to ia$, so we conclude that Eq. (2.17) holds for $\mathrm{Re}\, a \geq 0$.