# Recitation 5 (Oct. 27, 2017)

## 5.1 Gauge Fields as Connections

Throughout this section, we will work in natural units $\hbar = c = 1$.

Gauge symmetries play a crucial role in particle physics. As we mentioned in lecture, gauge fields (like the scalar and vector potentials of electromagnetism) allow us to describe a theory with nonlocal effects, such as the Aharanov–Bohm effect, in a local formulation.

Another viewpoint is that the addition of gauge fields allows us to turn global symmetries into local ones. This approach is somewhat the reverse direction of the logic that we used in Section 12.2.3 of the lecture notes, and it draws a connection to the language of general relativity that you may have seen previously.

Say we begin with a free particle. Its wavefunction $\psi(\boldsymbol{x}, t)$ obeys Schrödinger's equation,

$$i\frac{\partial}{\partial t}\psi = \frac{(-i\nabla)^2}{2m}\psi\,. \tag{5.1}$$

We know that all physical quantities are invariant under global phase rotations of the wavefunction,

$$\psi \to \psi' = e^{i\alpha}\psi\,. \tag{5.2}$$

This is a global symmetry of the system.

What happens if we try to make this into a local symmetry by promoting the parameter $\alpha$ to a function of space and time, $\alpha(\boldsymbol{x}, t)$? This is called *gauging* the symmetry. Using

$$\psi \to \psi' = e^{i\alpha(\boldsymbol{x},t)}\psi\,, \tag{5.3}$$

we can rewrite the Schrödinger equation (5.1) as

$$\left(i\frac{\partial}{\partial t} + \frac{\partial\alpha}{\partial t}\right)\psi' = \frac{(-i\nabla - \nabla\alpha)^2}{2m}\psi'\,. \tag{5.4}$$

We see that the transformed wave function no longer obeys the Schrödinger equation.

Another way of phrasing this problem is to note that if we demand that our theory satisfy the local symmetry in Eq. (5.3), then ordinary derivatives are no longer well-defined. It is convenient to use the four-vector notation that you may be familiar with from relativity (we will use the $+---$ convention). Recall the four-gradient,

$$\partial_\mu = \left(\frac{\partial}{\partial t}, \nabla\right), \quad \mu = 0, 1, 2, 3\,. \tag{5.5}$$

For our purposes here, you can just think of this as a notationally convenient way to package temporal and spatial derivatives. The ordinary directional derivative of the wavefunction $\psi$ in the $n^\mu$ direction is

$$n^\mu\partial_\mu\psi = \lim_{\epsilon\to 0}\frac{\psi(x^\mu + \epsilon n^\mu) - \psi(x^\mu)}{\epsilon}\,. \tag{5.6}$$

We see that if we can rotate the phase of the wavefunction arbitrarily at each point in spacetime, the numerator of this expression is no longer well-defined (even up to a global phase).

The solution to this problem is to define a new derivative, the *covariant derivative* $D_\mu$. This new derivative should have the property that under the gauge transformation, it transforms such that

$$D_\mu\psi \to e^{i\alpha(x)}D_\mu\psi\,. \tag{5.7}$$

Note that we are now suppressing the Lorentz indices of the coordinates, denoting $x^\mu$ simply by $x$, not to be confused with the spatial vector $\boldsymbol{x}$; we will use this notation throughout the rest of this section.

In order to define the covariant derivative, we assume that we have an object $U(y,x)$, called the *comparator*, that transforms under the gauge transformation as

$$U(y,x) \to e^{i\alpha(y)} U(y,x) e^{-i\alpha(x)} \,. \tag{5.8}$$

We are interested in such an object because when it is multiplied against $\psi(x)$, the product transforms as

$$U(y,x)\psi(x) \to e^{i\alpha(y)} U(y,x)\psi(x) \,. \tag{5.9}$$

That is, $U(y,x)\psi(x)$ transforms in the same way as $\psi(y)$. In a sense, $U(y,x)$ parallel transports gauge transformations from $x$ to $y$. Since $\psi(y)$ and $U(y,x)\psi(x)$ transform the same way, we can now directly compare them, and their difference is well-defined. Thus, we can define the new derivative as

$$n^\mu D_\mu \psi = \lim_{\epsilon\to 0} \frac{\psi(x+n\epsilon) - U(x+n\epsilon, x)\psi(x)}{\epsilon} \,. \tag{5.10}$$

By construction, this new derivative transforms just by an overall phase, as desired.

We must still construct the comparator $U(y,x)$. To define the derivative, we only need to define the comparator infinitesimally. We note that $U(y,x)$ must be a phase, because the system is only invariant under arbitrary phase changes; any other transformation would change the physical properties of the system. We can expand the comparator as

$$U(x+n\epsilon, x) = 1 - i\epsilon e n^\mu A_\mu(x) + O(\epsilon^2) \,, \tag{5.11}$$

where $e$ is a constant and the real vector $A_\mu(x)$ is the first derivative of $U(x+n\epsilon, x)$ (after pulling out the constant $e$). We now see that in order to parallel transport the gauge transformations, we must introduce a new vector field $A_\mu(x)$, called the *gauge field*.

In order for Eq. (5.11) to have the proper transformation properties of $U(y,x)$ given in Eq. (5.8), we find that the gauge field must transform as

$$A_\mu \to A_\mu - \frac{1}{e}\partial_\mu \alpha(x) \tag{5.12}$$

under the gauge transformation. Now the covariant derivative (5.10) can be written as

$$D_\mu \psi = \partial_\mu \psi + ieA_\mu \psi. \tag{5.13}$$

We see that this does indeed yield

$$D_\mu \psi \to e^{i\alpha(x)} D_\mu \psi. \tag{5.14}$$

Let's unpackage the pieces of the four-vector gauge field $A_\mu(x)$, writing it in the form

$$A^\mu = (\phi, \boldsymbol{A}) \,. \tag{5.15}$$

Here, $\phi$ and $\boldsymbol{A}$ are the familiar scalar and vector potentials from electromagnetism. The gauge transformation (5.12) of the gauge field then tells us that these must transform as

$$\begin{aligned} \phi \to \phi' &= \phi - \frac{1}{e}\frac{\partial \alpha}{\partial t} \,, \\ \boldsymbol{A} \to \boldsymbol{A}' &= \boldsymbol{A} + \frac{1}{e}\nabla\alpha \,. \end{aligned} \tag{5.16}$$

These match the forms of the gauge transformations we saw in lecture.

Furthermore, the form of the covariant derivative (5.13) tells us that the ordinary temporal and spatial derivatives should be replaced by

$$\begin{aligned} \frac{\partial}{\partial t} &\to \frac{\partial}{\partial t} + ie\phi\,, \\ \nabla &\to \nabla - ie\boldsymbol{A}\,. \end{aligned} \tag{5.17}$$

The proper form of the Schrödinger equation is then

$$\left(i\frac{\partial}{\partial t} - e\phi\right)\psi = \frac{(-i\nabla - e\boldsymbol{A})^2}{2m}\psi \tag{5.18}$$

In order to achieve this form for the Schrödinger equation, we must add terms to the Hamiltonian; we find that we are able to reproduce Eq. (5.18) with the Hamiltonian

$$H = \frac{(\boldsymbol{p} - e\boldsymbol{A})^2}{2m} + e\phi\,. \tag{5.19}$$

This is the familiar Hamiltonian for a charged particle moving in an electromagnetic field; a Legendre transform yields the corresponding Lagrangian,

$$L = \frac{1}{2}m\dot{\boldsymbol{x}}^2 + e\boldsymbol{A}\cdot\dot{\boldsymbol{x}} - e\phi\,. \tag{5.20}$$

We see that by taking this perspective, we were able to derive the Lagrangian for a charged particle moving in an electromagnetic field; this form was necessary in order to promote the global symmetry under phase rotations to a local one. The relation of gauge symmetries to general relativity (which is, after all, where these symmetries got the name "gauge") also becomes clear from this viewpoint: the comparator $U(y, x)$ is the *parallel transport* necessary to relate the wavefunction at two different points of spacetime, and $A_\mu$ is the corresponding *connection* (similar to the Levi–Civita connection). Just as in general relativity, the covariant derivative $D_\mu$ contains a term with the ordinary derivative and a term with the connection. In this language, the field strength tensor $F_{\mu\nu}$ is the *curvature* corresponding to the connection $A_\mu$, analogous to the Riemann curvature tensor $R^\rho{}_{\sigma\mu\nu}$ of general relativity. The formal mathematical structure underlying these concepts, both in electromagnetism and in general relativity, is called a *fiber bundle*.