8.323 Relativistic Quantum Field Theory I
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8.323: Relativistic Quantum Field Theory I

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LECTURE NOTES 2
NOTES ON THE EULER-MACLAURIN SUMMATION FORMULA

These notes are intended to supplement the Casimir effect problem of Problem Set 3 (2008). That calculation depended crucially on the Euler-Maclaurin summation formula, which was stated without derivation. Here I will give a self-contained derivation of the Euler-Maclaurin formula. For pedagogical reasons I will first derive the formula without any reference to Bernoulli numbers, and afterward I will show that the answer can be expressed in terms of these numbers. An explicit expression will be obtained for the remainder that survives after a finite number of terms in the series are summed, and in an optional appendix I will show how to simplify this remainder to obtain the form given by Abramowitz and Stegun.

The Euler-Maclaurin formula relates the sum of a function evaluated at evenly spaced points to the corresponding integral approximation, providing a systematic method of calculating corrections in terms of the derivatives of the function evaluated at the endpoints. Consider first a function defined on the interval $-1 \leq x \leq 1$, for which we can imagine approximating the sum of $f(-1) + f(1)$ by the integral of the function over the interval:

$$f(-1) + f(1) = \int_{-1}^{1} \mathrm{d}x\, f(x) + R_1 \ , \tag{1}$$

where R_1 represents a correction term that we want to understand. One can find an exact expression for R_1 by applying an integration by parts to the integral:

$$\int_{-1}^{1} \mathrm{d}x\, f(x) = f(-1) + f(1) - \int_{-1}^{1} \mathrm{d}x\, x\, f'(x) \ , \tag{2}$$

so

$$R_1 = \int_{-1}^{1} \mathrm{d}x\, x\, f'(x) \ , \tag{3}$$

where a prime denotes a derivative with respect to x.

1. Expansion by successive integrations by parts:

We want an approximation that is useful for smooth functions $f(x)$, and a smooth function is one for which the higher derivatives tend to be small. Therefore, if we can extract more terms in a way that leaves a remainder term that depends only on high derivatives of the function, then we have made progress. This can be accomplished by successively integrating by parts, each time differentiating $f(x)$ and integrating the function that multiplies it. We can define a set of functions

$$V_0(x) \equiv 1 \ , \ \ V_1(x) \equiv x \ , \tag{4}$$

and

$$V_n(x) \equiv \int \mathrm{d}x \, V_{n-1}(x) \ . \tag{5}$$

Eq. (5) is not quite well-defined, however, because each indefinite integral is defined only up to an arbitrary constant of integration. Regardless of how these constants of integration are chosen, however, one can rewrite Eq. (1) by using Eq. (3) and then successively integrating by parts:

$$\begin{aligned} f(-1)+f(1) &= \int_{-1}^{1} \mathrm{d}x \, f(x) + \int_{-1}^{1} \mathrm{d}x \, V_1(x) \, f'(x) \\ &= \int_{-1}^{1} \mathrm{d}x \, f(x) + V_2(1) f'(1) - V_2(-1) f'(-1) - \int_{-1}^{1} \mathrm{d}x \, V_2(x) \, f''(x) \\ &= \int_{-1}^{1} \mathrm{d}x \, f(x) + \Big[V_2(1) f'(1) - V_2(-1) f'(-1) \Big] \\ &\quad - \Big[V_3(1) f''(1) - V_3(-1) f''(-1) \Big] + \Big[V_4(1) f'''(1) - V_4(-1) f'''(-1) \Big] + \ldots \\ &\quad + \Big[V_{2n}(1) f^{2n-1}(1) - V_{2n}(-1) f^{2n-1}(-1) \Big] - \int_{-1}^{1} \mathrm{d}x \, V_{2n}(x) \, f^{2n}(x) \\ &= \int_{-1}^{1} \mathrm{d}x \, f(x) + \sum_{\ell=2}^{2n} (-1)^{\ell} \Big[V_{\ell}(1) \, f^{\ell-1}(1) - V_{\ell}(-1) \, f^{\ell-1}(-1) \Big] \\ &\qquad - \int_{-1}^{1} \mathrm{d}x \, V_{2n}(x) \, f^{2n}(x) \ , \end{aligned} \tag{6}$$

where $f^n(x)$ denotes the n^{th} derivative of f with respect to x.

2. Elimination of the odd ℓ contributions:

Eq. (6) is valid for any choice of integration constants in Eq. (5), so we can seek a choice that simplifies the result. Note that $V_1(x)$ is odd under $x \to -x$. We can therefore choose the integration constants so that

$$V_n(-x) = \begin{cases} V_n(x) & \text{if } n \text{ is even} \\ -V_n(x) & \text{if } n \text{ is odd} \quad . \end{cases} \tag{7}$$

This even/odd requirement uniquely fixes the integration constant in Eq. (5) when n is odd, because the sum of an odd function and a constant would no longer be odd. We are still free, however, to choose the integration constants when n is even.

Using the even/odd property, Eq. (6) can be simplified to

$$\begin{aligned} f(-1) + f(1) = \int_{-1}^{1} \mathrm{d}x\, f(x) + \sum_{\ell=2}^{2n} (-1)^{\ell} V_\ell(1) \left[f^{\ell-1}(1) - (-1)^{\ell} f^{\ell-1}(-1) \right] \\ - \int_{-1}^{1} \mathrm{d}x\, V_{2n}(x)\, f^{2n}(x) \ . \end{aligned} \tag{8}$$

Note that the terms in $V_\ell(1)$ for even ℓ involve the difference of $f^{\ell-1}$ at the two endpoints, while the terms for odd ℓ involve the sum. Eq. (8) describes a single interval, however, and our goal is to obtain a formula valid for any number of intervals. We will do this by first generalizing Eq. (8) to apply to an arbitrary interval $a \le x \le a+h$, and then applying it to each interval in a succession of evenly spaced intervals. When this succession is summed, the even ℓ terms involving the differences of the endpoints will cancel at each interior point, but the odd ℓ terms will add. The odd ℓ terms can therefore make a considerably more complicated contribution to the answer, but we can force them to vanish by using the remaining freedom in the choice of integration constants. When n is even in Eq. (5), we choose the integration constant so that

$$\int_{-1}^{1} \mathrm{d}x\, V_n(x) \equiv 0 \ . \tag{9}$$

Eq. (9) is always true for odd functions, so it is true for all $n > 0$. It then follows for all $n > 1$ that

$$V_n(1) - V_n(-1) = \int_{-1}^{1} \mathrm{d}x\, V_{n-1}(x) = 0 \ . \tag{10}$$

If n is odd then Eq. (7) implies that $V_n(-1) = -V_n(1)$, and so

$$V_n(1) = V_n(-1) = 0 \qquad \text{for all odd } n > 1 \ , \tag{11}$$

as desired.

The V_n's are now uniquely defined. In our construction we used the antisymmetry property of Eq. (5) to fix the constant of integration for odd n, and the vanishing of the integral in Eq. (9) to fix the integration constant for even n. Eq. (9), however, holds also for odd n, and is sufficient to fix the integration constant for the odd n cases. The functions $V_n(x)$ can therefore be defined succinctly by

$$V_0(x) \equiv 1 \ , \tag{12a}$$

$$V_n(x) \equiv \int \mathrm{d}x \, V_{n-1}(x) \ , \text{ and} \tag{12b}$$

$$\int_{-1}^{1} \mathrm{d}x \, V_n(x) \equiv 0 \qquad (\text{for } n > 0) \ . \tag{12c}$$

We can use these properties to build a table for the lowest values of n:

$\boldsymbol{n}$	$\boldsymbol{V_n(x)}$	$\boldsymbol{V_n(1)}$
0	$V_0(x) = 1$	1
1	$V_1(x) = x$	1
2	$V_2(x) = \dfrac{x^2}{2} - \dfrac{1}{6}$	$\dfrac{1}{3}$
3	$V_3(x) = \dfrac{x^3}{6} - \dfrac{x}{6}$	0
4	$V_4(x) = \dfrac{x^4}{24} - \dfrac{x^2}{12} + \dfrac{7}{360}$	$-\dfrac{1}{45}$
5	$V_5(x) = \dfrac{x^5}{120} - \dfrac{x^3}{36} + \dfrac{7x}{360}$	0
6	$V_6(x) = \dfrac{x^6}{720} - \dfrac{x^4}{144} + \dfrac{7x^2}{720} - \dfrac{31}{15120}$	$\dfrac{2}{945}$

(13)

Eq. (11) guarantees that only the even-ℓ terms contribute to Eq. (8), so we can set $\ell = 2k$ and rewrite Eq. (8) as

$$
\begin{aligned}
f(-1) + f(1) = \int_{-1}^{1} \mathrm{d}x\, f(x) + \sum_{k=1}^{n} V_{2k}(1) \Big[f^{2k-1}(1) - f^{2k-1}(-1) \Big] \\
- \int_{-1}^{1} \mathrm{d}x\, V_{2n}(x)\, f^{2n}(x) \ .
\end{aligned}
\tag{14}
$$

3. Application to an arbitrary interval:

To apply Eq. (14) to an arbitrary interval $a < x < a + h$, one needs only to change variables. Let $f(x) = \tilde{f}(\tilde{x})$, where

$$
\tilde{x} = (x+1)\frac{h}{2} + a \ . \tag{15}
$$

Rewriting Eq. (14) in terms of $\tilde{f}(\tilde{x})$, while dividing the whole equation by 2 for later convenience, one has

$$
\begin{aligned}
\frac{1}{2}\big[\tilde{f}(a) + \tilde{f}(a+h)\big] = {} & \frac{1}{h} \int_{a}^{a+h} \mathrm{d}\tilde{x}\, \tilde{f}(\tilde{x}) \\
& + \frac{1}{2} \sum_{k=1}^{n} \left(\frac{h}{2}\right)^{2k-1} V_{2k}(1) \Big[\tilde{f}^{2k-1}(a+h) - \tilde{f}^{2k-1}(a) \Big] \\
& - \frac{1}{2} \left(\frac{h}{2}\right)^{2n-1} \int_{a}^{a+h} \mathrm{d}\tilde{x}\, V_{2n}\left(\frac{2}{h}(\tilde{x} - a) - 1\right) \tilde{f}^{2n}(\tilde{x}) \ ,
\end{aligned}
\tag{16}
$$

where $\tilde{f}^{n}(\tilde{x})$ denotes the n'th derivative of $\tilde{f}$ with respect to its argument $\tilde{x}$. Now that the original f and x have been eliminated, we can drop the tilde superscripts that appear in Eq. (16).

4. Application to an arbitrary sum of intervals:

The problem can now be completed by extending Eq. (16) to an interval $a < x < b$, divided into m steps of size $h = (b - a)/m$. Adding an expression of the form

(16) for each interval of size h, one has

$$\boxed{\begin{aligned}\sum_{k=0}^{m} f(a+kh) = \frac{1}{h}\int_a^b \mathrm{d}x\, f(x) + \frac{1}{2}\Big[f(a)+f(b)\Big] \\ + \frac{1}{2}\sum_{k=1}^{n}\left(\frac{h}{2}\right)^{2k-1} V_{2k}(1)\Big[f^{2k-1}(b) - f^{2k-1}(a)\Big] \\ - \frac{1}{2}\left(\frac{h}{2}\right)^{2n-1}\int_a^{a+h}\mathrm{d}x\, V_{2n}\left(\frac{2}{h}(x-a)-1\right)\sum_{k=0}^{m-1} f^{2n}(x+kh)\ .\end{aligned}} \tag{17}$$

Note that when one adds up the left-hand sides of the expressions of the form (16), all the terms have coefficient 1 except for the first and last term, each of which have coefficient $\frac{1}{2}$. In Eq. (17), the sum is written with all terms having coefficient 1, and the correction for the first and last term has been moved to the right-hand side.

For all practical purposes, including the application to the Casimir effect in Problem Set 4, Eq. (17) is all that is necessary. For the Casimir application $a = 0$, $b = \infty$, and $h = 1$, so Eq. (17) becomes

$$\boxed{\begin{aligned}\sum_{k=0}^{\infty} f(k) = \int_0^{\infty}\mathrm{d}x\, f(x) + \frac{1}{2}f(0) - \frac{1}{12}f'(0) + \frac{1}{720}f'''(0) \\ - \frac{1}{30,240}f'''''(0) + \dots\ .\end{aligned}} \tag{18}$$

On the problem set the sum on the left started at 1 instead of 0, so the coefficient of $f(0)$ on the right was $-\frac{1}{2}$ instead of $\frac{1}{2}$. (Note that the coefficient of the $f(0)$ term was printed incorrectly in Huang's book and in 8.323 problem sets from some years before 2003.)

Before leaving the subject, however, one might want to establish the connection between Eq. (17) and the usual expression of the Euler-Maclaurin formula in terms of Bernoulli numbers, and one might wish to find a cleaner way to express the final term of Eq. (17), often called the remainder term. One normally does not evaluate this term, but one wants to use it to argue that the remainder is small.

5. Connection to the Bernoulli numbers:

The Euler-Maclaurin summation formula is stated in Abramowitz and Stegun,* hereafter called A&S, as follows:

> Let $F(x)$ have its first $2n$ derivatives continuous on an interval (a,b). Divide the interval into m equal parts and let $h=(b-a)/m$. Then for some θ, $1>\theta>0$, depending on $F^{(2n)}(x)$ on (a,b), we have

$$\boxed{\begin{aligned}\sum_{k=0}^{m} F(a+kh) = \frac{1}{h}\int_a^b F(t)\,\mathrm{d}t + \frac{1}{2}\{F(b)+F(a)\} \\ + \sum_{k=1}^{n-1}\frac{h^{2k-1}}{(2k)!}B_{2k}\{F^{(2k-1)}(b)-F^{(2k-1)}(a)\} \\ + \frac{h^{2n}}{(2n)!}B_{2n}\sum_{k=0}^{m-1}F^{(2n)}(a+kh+\theta h)\ .\end{aligned}} \tag{19}$$

Disregarding for now the remainder term (the 3rd term on the right), Eq. (19) agrees with Eq. (17) provided that

$$\frac{B_{2k}}{(2k)!} = \frac{V_{2k}(1)}{2^{2k}}\ . \tag{20}$$

According to A&S, p. 804, the Bernoulli numbers B_n are defined in terms of the Bernoulli polynomials $B_n(x)$, which are defined by the generating function

$$\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \qquad \text{(for } t<2\pi\text{)}. \tag{21}$$

The Bernoulli numbers are given by

$$B_n = B_n(0)\ . \tag{22}$$

From these relations one can easily (i.e., easily with the help of a computer algebra program) calculate the lowest Bernoulli polynomials:

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* M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions, With Formulas, Graphs, and Mathematical Tables*, p. 806.

n	$B_n(x)$	$B_n = B_n(0)$
0	$B_0(x) = 1$	1
1	$B_1(x) = x - \frac{1}{2}$	$-\frac{1}{2}$
2	$B_2(x) = x^2 - x + \frac{1}{6}$	$\frac{1}{6}$
3	$B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$	0
4	$B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}$	$-\frac{1}{30}$
5	$B_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x$	0
6	$B_6(x) = x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2 + \frac{1}{42}$	$\frac{1}{42}$

(23)

By comparing Eqs. (23) with Eqs. (13), one can conjecture the equality

$$V_n(x) = \tilde{V}_n(x) \ , \tag{24}$$

where

$$\tilde{V}_n(x) \equiv \frac{2^n}{n!} B_n\left(\frac{x+1}{2}\right) \ . \tag{25}$$

To prove this equality, it is sufficient to verify that $\tilde{V}_n(x)$ satisfies the relations (12), since these were the relations that defined $V_n(x)$. It is straightforward to determine the properties required for $B_n(x)$ so that $\tilde{V}_n(x)$ obeys each of the relations (12):

$$B_0(x) = 1 \ , \tag{26a}$$

$$B_n(x) = n \int \mathrm{d}x \, B_{n-1}(x) \ , \text{ and} \tag{26b}$$

$$\int_0^1 \mathrm{d}x \, B_n(x) = 0 \qquad (\text{for } n > 0) \ . \tag{26c}$$

Eq. (26a) was already written within Eqs. (23). To verify Eq. (26b), differentiate the generating equation (21) with respect to x:

$$\frac{t^2 e^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{\mathrm{d}B_n(x)}{\mathrm{d}x} \frac{t^n}{n!} \; . \tag{27}$$

One can obtain another expansion of the same quantity by multiplying the generating equation by t:

$$\begin{aligned} \frac{t^2 e^{xt}}{e^t - 1} &= \sum_{n=0}^{\infty} B_n(x) \frac{t^{n+1}}{n!} \\ &= \sum_{n=1}^{\infty} B_{n-1} \frac{t^n}{(n-1)!} \; . \end{aligned} \tag{28}$$

Comparing like powers of t in expansions (27) and (28), one finds

$$\frac{\mathrm{d}B_n}{\mathrm{d}x} = n\, B_{n-1}(x) \; , \tag{29}$$

which is equivalent to Eq. (26b). Finally, to verify Eq. (26c), one can integrate the generating equation over x from 0 to 1:

$$\int_0^1 \mathrm{d}x \, \frac{t e^{xt}}{e^t - 1} = 1 = \sum_{n=0}^{\infty} \frac{t^n}{n!} \int_0^1 \mathrm{d}x \, B_n(x) \; . \tag{30}$$

Again by comparing powers of t, one verifies Eq. (26c). Thus $\tilde{V}_n(x)$ obeys all of the relations (12), and hence $V_n(x) = \tilde{V}_n(x)$.

Having established Eq. (24), it follows immediately that

$$V_{2k}(1) = V_{2k}(-1) = \frac{2^{2k}}{(2k)!} B_{2k}(0) \quad , \tag{31}$$

which is just what is needed to verify Eq. (20), and hence the agreement of our series expansion with that of A&S.

Appendix: Simplification of the remainder term:

The remainder term is the final term on the right-hand side of Eq. (17), given by

$$R = -\frac{1}{2} \left(\frac{h}{2}\right)^{2n-1} \int_a^{a+h} \mathrm{d}x \, V_{2n}\left(\frac{2}{h}(x-a) - 1\right) \sum_{k=0}^{m-1} f^{2n}(x + kh) \; . \tag{32}$$

The goal of this appendix is to manipulate the remainder into the form given by A&S, as shown in Eq. (19) of this document. I am including this appendix for completeness, but it is not physically important and you need not read it if you are not curious.

First, notice that our definition of n is different from A&S's, since the sum on the right-hand side of our Eq. (17) extends up to n, while the sum in A&S's equation (Eq. (19)) extends only up to $n-1$. Thus, if our remainder term is to agree with A&S's, it should be expressed in terms of f^{2n+2}, not f^{2n} as in Eq. (32). To accomplish this change, we will integrate by parts twice. The surface term vanishes for the first integration by parts, since V_{2n+1} has an odd subscript and therefore vanishes at the endpoints according to Eq. (11). For the second integration by parts there is a surface term which must be kept. In detail, the two integrations by parts yield

$$R = \frac{1}{2}\left(\frac{h}{2}\right)^{2n} \int_a^{a+h} \mathrm{d}x \, V_{2n+1}\left(\frac{2}{h}(x-a)-1\right) \sum_{k=0}^{m-1} f^{2n+1}(x+kh) \tag{33a}$$

$$\begin{aligned} &= \frac{1}{2}\left(\frac{h}{2}\right)^{2n+1} V_{2n+2}(1) \sum_{k=0}^{m-1} \Big[f^{2n+1}\big(a+(k+1)h\big) - f^{2n+1}(a+kh) \Big] \\ &\quad - \frac{1}{2}\left(\frac{h}{2}\right)^{2n+1} \int_a^{a+h} \mathrm{d}x \, V_{2n+2}\left(\frac{2}{h}(x-a)-1\right) \sum_{k=0}^{m-1} f^{2n+2}(x+kh) \end{aligned} \tag{33b}$$

Now notice that the first term on the right-hand side of Eq. (33b) can be rewritten using

$$f^{2n+1}\big(a+(k+1)h\big) - f^{2n+1}(a+kh) = \int_a^{a+h} \mathrm{d}x \, f^{2n+2}(x+kh) \ , \tag{34}$$

which allows one to combine the two terms:

$$R = \frac{1}{2}\left(\frac{h}{2}\right)^{2n+1} V_{2n+2}(1) \int_a^{a+h} \mathrm{d}x \, w(x)\, G(x) \ , \tag{35}$$

where

$$w(x) = 1 - \frac{V_{2n+2}\left(\frac{2}{h}(x-a)-1\right)}{V_{2n+2}(1)} \tag{36}$$

and

$$G(x) = \sum_{k=0}^{m-1} f^{2n+2}(x+kh) \ . \tag{37}$$

Using Eqs. (24), (25) and (31) to rewrite this expression in terms of B_{2n+2}, one finds

$$R = h^{2n+1} \frac{B_{2n+2}}{(2n+2)!} \int_a^{a+h} \mathrm{d}x \, w(x) \, G(x) \ , \tag{38}$$

where

$$w(x) = 1 - \frac{B_{2n+2}\left(\frac{x-a}{h}\right)}{B_{2n+2}} \ . \tag{39}$$

To complete the argument, one needs to use the fact that

$$|B_{2n}| > |B_{2n}(x)| \qquad (\text{for } n = 1, 2, \ldots, \text{ and } 0 < x < 1) \ , \tag{40}$$

which we will prove below. This implies that the second term in Eq. (39) has magnitude less than 1, and hence

$$w(x) > 0 \ . \tag{41}$$

Furthermore, from Eq. (26c) the second term in Eq. (39) vanishes when integrated over x from a to $a + h$, so

$$\int_a^{a+h} \mathrm{d}x \, w(x) = h \ . \tag{42}$$

Eqs. (41) and (42) imply that we can interpret $w(x)$ as a weight factor in the computation of a weighted average, with

$$\big\langle G(x) \big\rangle \equiv \frac{1}{h} \int_a^{a+h} \mathrm{d}x \, w(x) \, G(x) \ . \tag{43}$$

If we assume that every term on the right-hand side of Eq. (37) is continuous, then $G(x)$ is continuous, and we can apply the mean value theorem* to conclude that, somewhere in the range of integration ($a < x < a + h$), $G(x)$ must be equal to its mean value $\big\langle G(x) \big\rangle$. Thus there exists some number θ in the range $0 < \theta < 1$ such that

$$G(a + \theta h) = \big\langle G(x) \big\rangle = \frac{1}{h} \int_a^{a+h} \mathrm{d}x \, w(x) \, G(x) \ . \tag{44}$$

Using the above relation to replace the integral in Eq. (38), one has finally

$$\begin{aligned} R &= h^{2n+2} \frac{B_{2n+2}}{(2n+2)!} G(a + \theta h) \\ &= h^{2n+2} \frac{B_{2n+2}}{(2n+2)!} \sum_{k=0}^{m-1} f^{2n+2}(a + kh + \theta h) \ , \end{aligned} \tag{45}$$

* See, for example, *Methods of Mathematical Physics,* Third Edition, by Sir Harold Jeffreys and Bertha Swirles (Lady Jeffreys), Cambridge University Press, 1962, p. 50, section 1.132.

which matches the remainder term in A&S's equation (Eq. (19)).

We have reached the end, but to complete the proof we must justify the inequality (40). This inequality was necessary to assure the positivity of $w(x)$, which in turn was necessary for the mean value theorem.

The only proof that I could construct for this inequality depends on showing that the general shape of the Bernoulli polynomials $B_n(x)$ in the interval $(0, 1)$, for $n \geq 3$, is always one of four possibilities, depending on n mod 4. Sample graphs illustrating these shapes are shown on the following page. Specifically,

***n* mod 4 = 0:** $B_n(x)$ is symmetric about $x = \frac{1}{2}$. The maximum is at $x = \frac{1}{2}$, where $B_n > 0$, and the minimum is at $x = 0$ and $x = 1$, where $B_n < 0$. The slope is negative for $\frac{1}{2} < x < 1$, and vanishes at the endpoints of this region.

***n* mod 4 = 1:** $B_n(x)$ is antisymmetric about $x = \frac{1}{2}$, and vanishes at $x = 0$ and $x = 1$. Between $x = \frac{1}{2}$ and $x = 1$ the function rises monotonically to a maximum and then falls monotonically.

***n* mod 4 = 2:** $B_n(x)$ is symmetric about $x = \frac{1}{2}$. The minimum is at $x = \frac{1}{2}$, where $B_n < 0$, and the maximum is at $x = 0$ and $x = 1$, where $B_n > 0$. The slope is positive for $\frac{1}{2} < x < 1$, and vanishes at the endpoints of this region.

***n* mod 4 = 3:** $B_n(x)$ is antisymmetric about $x = \frac{1}{2}$, and vanishes at $x = 0$ and $x = 1$. Between $x = \frac{1}{2}$ and $x = 1$ the function falls monotonically to a minimum and then rises monotonically.

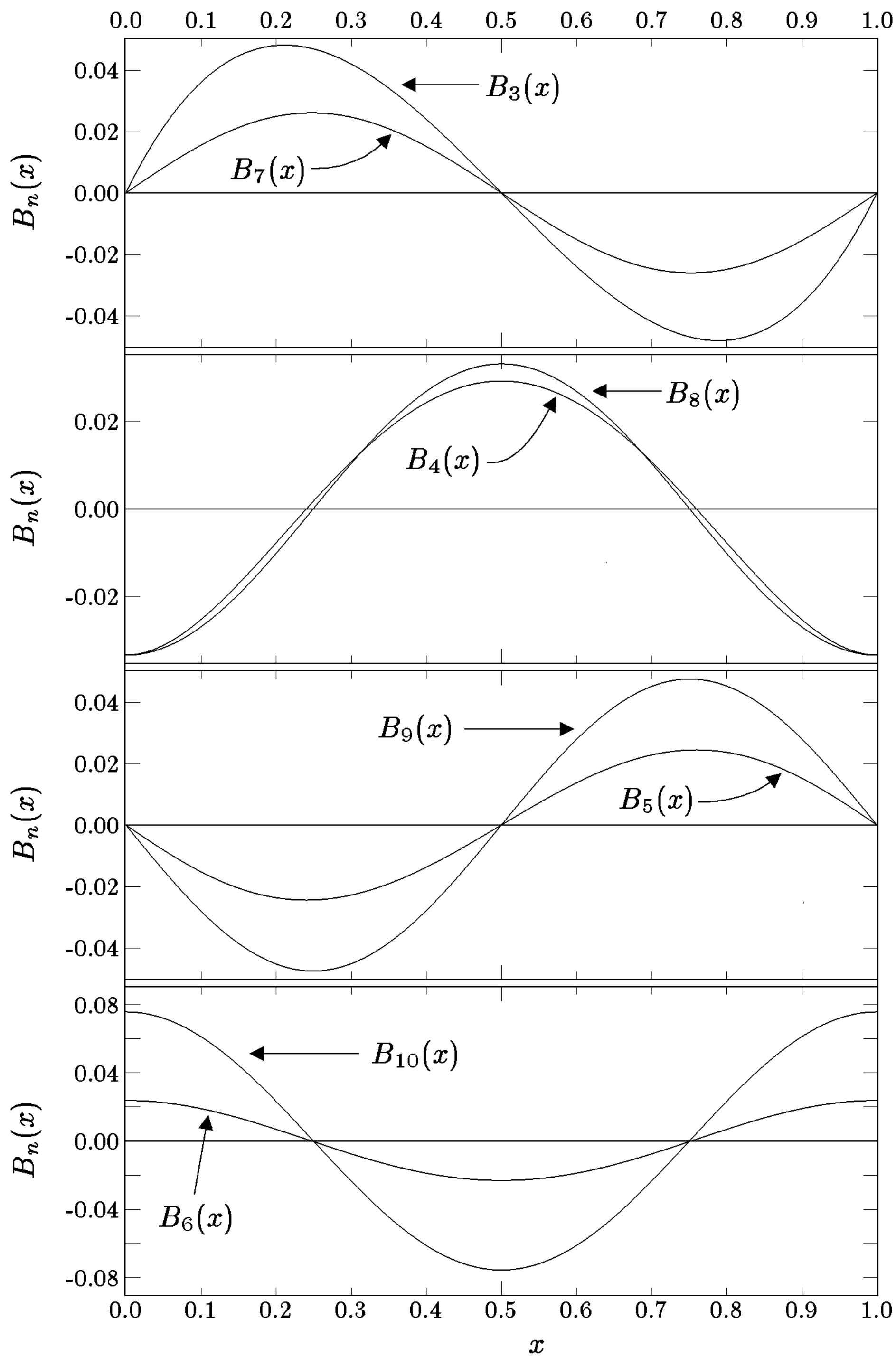
Note that we have already shown (by Eqs. (11), (24), and (25)) that $B_n(x)$ vanishes at 0 and 1 for n odd. The remaining properties listed above can be shown by induction: one verifies that $B_3(x)$ is being correctly described, and then one uses Eqs. (26b) and (26c) to show that the properties for each value of n mod 4 imply the properties for $(n + 1)$ mod 4.

(Note that graphically it appears that the zeros of $B_6(x)$ and $B_{10}(x)$ coincide, but this is not exactly true. The zeros of $B_6(x)$, $B_{10}(x)$, and $B_{14}(x)$ lie at 0.2475407, 0.2498447, and 0.2499903, respectively.)

From the description above for $B_n(x)$ when n is even, one can see that the maximum absolute value of the function in the range $0 < x < 1$ must occur either at the endpoints or at $x = \frac{1}{2}$. We can determine which of these two it is by using the generating function (21) to derive an identity that relates them.

Consider the generating equation (21) for $x = 0$, but with t replaced by $t/2$:

$$\sum_{n=0}^{\infty} B_n(0) \frac{\left(\frac{t}{2}\right)^n}{n!} = \frac{1}{2} \frac{t}{e^{(t/2)} - 1} \ . \tag{46}$$



Bernoulli Polynomials $B_n(x)$

By manipulating the right-hand side, it can be re-expressed in terms of Bernoulli functions by using the generating equation:

$$\begin{aligned}
\sum_{n=0}^{\infty} B_n(0) \frac{\left(\frac{t}{2}\right)^n}{n!} &= \frac{1}{2} \frac{t}{e^{(t/2)} - 1} \cdot \frac{e^{(t/2)} + 1}{e^{(t/2)} + 1} \\
&= \frac{1}{2} \frac{te^{(t/2)} + t}{e^t - 1} \\
&= \frac{1}{2} \left\{ \sum_{n=0}^{\infty} B_n\left(\tfrac{1}{2}\right) \frac{t^n}{n!} + \sum_{n=0}^{\infty} B_n(0) \frac{t^n}{n!} \right\} \ .
\end{aligned} \tag{47}$$

Comparing like powers of t on both sides of the equation, one finds

$$\frac{B_n(0)}{2^n} = \frac{1}{2} \left\{ B_n\left(\tfrac{1}{2}\right) + B_n(0) \right\} \ , \tag{48}$$

which implies that

$$B_n\left(\tfrac{1}{2}\right) = -B_n(0) \left[1 - \frac{1}{2^{n-1}} \right] \ . \tag{49}$$

It follows that $|B_n(0)| > \left|B_n\left(\tfrac{1}{2}\right)\right|$, and therefore Eq. (40) holds. Our proof excluded the special case $B_2(x)$, which differs from the cases of larger n in that its derivative does not vanish at 0 and 1. It can easily be verified, however, that the maximum absolute value of $B_2(x)$ for $0 < x < 1$ must occur at $x = \frac{1}{2}$ or at the endpoints, so Eq. (49) can again be used to show that Eq. (40) applies in this case as well.