Problem Set 2 Solutions

1. (a) **PS 15.1 (c) and (d):** For this problem it is very useful to note that $t^1 - t^7$ are Pauli matrices in different subspaces. t^8 can than be treated separately. We would like to verify that

$$\mathrm{tr}\left[t_r^a t_r^b\right] = C(r)\delta^{ab} \tag{1}$$

and evaluate $C(r)$ for the fundamental representation of $SU(3)$. Just checking a few cases, for example

$$t^1 t^2 = \frac{1}{4}\begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2}$$

and so

$$\mathrm{tr}\left[t^1 t^2\right] = 0 \tag{3}$$

which is a check on (1). Similarly

$$\mathrm{tr}\left[t^1 t^1\right] = \mathrm{tr}\left[\frac{1}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] = \frac{1}{2} \tag{4}$$

implying that $C(r) = \frac{1}{2}$ for the fundamental of $SU(3)$. All other cases of (1) can be checked in a similar way.

We now wish to verify that

$$t_r^a t_r^a = C_2(r) \cdot \mathbb{1} \tag{5}$$

A short but somewhat tedious computation gives $C_2(r) = 4/3$. Thus the relation

$$d(r)C_2(r) = d(G)C(r) \tag{6}$$

$$3 \cdot \frac{4}{3} = 8 \cdot \frac{1}{2} \tag{7}$$

is verified.

(b) **PS 15.2:** The generators of the adjoint representation of $SU(2)$ are computed to be

$$(t^1)_{ab} = i\epsilon^{a1b} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \tag{8}$$

$$(t^2)_{ab} = i\epsilon^{a2b} = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \tag{9}$$

$$(t^3)_{ab} = i\epsilon^{a3b} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{10}$$

(you'll notice that these are also the generators of the fundamental representation of $SO(3)$.) We can compute $C(G)$ by, for example

$$\mathrm{tr}\left[t^1 t^1\right] = \mathrm{tr}\left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] = 2 \tag{11}$$

so $C(G) = 2$. One also finds that

$$(t^1)^2 + (t^2)^2 + (t^3)^2 = 2 \cdot \mathbb{1} \tag{12}$$

so that $C_2(G) = 2$ as well. These results agree with PS (15.104).

2. (a) The equations of motion for the gauge field are

$$D_\mu F^{\mu\nu a} = J^{\nu a} \tag{13}$$

with

$$J^{\nu a} = g\bar{\psi}\gamma^\nu T^a \psi \tag{14}$$

The equations of motion for the spinor and its conjugate are

$$\not{D}\psi = m\psi \qquad \partial_\mu\bar{\psi}\gamma^\mu + ig\bar{\psi}\not{A} = -m\bar{\psi} \ . \tag{15}$$

Glancing at (14) we can see that the current must be covariantly conserved. By virtue of (13)

$$D_\nu J^\nu = D_\nu D_\mu F^{\mu\nu} = \frac{1}{2}\left[D_\nu, D_\mu\right] F^{\mu\nu} + \frac{1}{2}\left\{D_\nu, D_\mu\right\} F^{\mu\nu} = \frac{1}{2}\left[D_\nu, D_\mu\right] F^{\mu\nu} \ , \tag{16}$$

where we used that $\{D_\nu, D_\mu\}$ is symmetric, while $F^{\mu\nu}$ is antisymmetric in $\mu,\ \nu$. Now we use the definition of the field strength tensor and pay attention to the fact that it acts on an object in the *adjoint representation*:

$$\frac{1}{2}\left[D_\nu, D_\mu\right] F^{\mu\nu} = -\frac{ig}{2}F^{(adj)}_{\nu\mu} \cdot F^{\mu\nu} = -\frac{ig}{2}\left[F_{\nu\mu}, F^{\mu\nu}\right] = \frac{ig}{2}\left[F_{\nu\mu}, F^{\mu\nu}\right] = 0 \ , \tag{17}$$

where for the penultimate equality we raised and lowered indices and used the antisymmetry of the commutator.
An alternate solution in components is:

$$D_\nu J^{a\nu} = \frac{1}{2}\left[D_\nu, D_\mu\right] F^{a\mu\nu} = \frac{ig}{2} f^{abc} F^b_{\mu\nu} F^{a\mu\nu} = 0 \ , \tag{18}$$

where we again used the action of the covariant derivative on an object in the *adjoint representation* and the total antisymmetry of f^{abc}.
We can also directly calculate

$$D_\nu J^{\nu a} = g(\partial_\nu\bar{\psi})\gamma^\nu T^a\psi + g\bar{\psi}T^a\not{\partial}\psi + g^2 f^{abc} A^b_\nu \bar{\psi}\gamma^\nu T^c\psi \tag{19}$$

The third term in this expression is equal to

$$g^2 f^{abc} A^b_\nu \bar{\psi}\gamma^\nu T^c\psi = -ig^2 A^b_\nu \bar{\psi}\gamma^\nu \left[T^a, T^b\right]\psi = -ig^2\left(\bar{\psi}T^a\not{A}\psi - \bar{\psi}\not{A}T^a\psi\right) \tag{20}$$

where A should be understood as a matrix determined by the matter representation. Therefore (19) is equal to

$$D_\nu J^{\nu a} = g\bar{\psi}T^a\not{D}\psi + g\left(\partial_\mu\bar{\psi}\gamma^\mu + ig\bar{\psi}\not{A}\right)T^a\psi = mg\bar{\psi}T^a\psi - mg\bar{\psi}T^a\psi = 0 \tag{21}$$

where in the second equality we have used (15).

(b) We seek to show that both sides of (13) transform in the same way. Now, we have shown in class that F transforms in the adjoint meaning that infinitesimally

$$\delta F^{\mu\nu a} = -f^{abc}\alpha^b F^{\mu\nu c} \tag{22}$$

The covariant derivative is constructed, almost by definition, so that the covariant derivative of some object has the same transformation property, namely that

$$\delta(D_\mu F^{\mu\nu a}) = -f^{abc}\alpha^b D_\mu F^{\mu\nu c} \tag{23}$$

Now, infintesimally $\delta\psi = i\alpha^b T_b\psi$, meaning

$$\begin{aligned}\delta J^{\nu a} &= i\alpha^b\left(-\bar{\psi}T^bT^a\gamma^\nu\psi + \bar{\psi}T^aT^b\gamma^\nu\psi\right) &(24)\\ &= i\alpha^b\bar{\psi}\left[T_a, T_b\right]\gamma^\nu\psi &(25)\\ &= -f^{abc}\alpha^b J^{\nu c} &(26)\end{aligned}$$

thus showing that both sides of equation (13) transform the same.

We can also, calculate the finite transformation of J^ν. Firstly, we derive the result using the abstract language and properties we learned from the notes by Prof. Zwiebach. From Zwi (27) the action of a gauge transformation is:

$$D_\mu F^{\mu\nu} \to \text{adj}U\left(D_\mu F^{\mu\nu}\right) = U\left(D_\mu F^{\mu\nu}\right)U^\dagger = \left(D_\mu F^{\mu\nu}\right)^a U T^r_a U^\dagger = \left(D_\mu F^{\mu\nu}\right)^a T^r_b D_{ba} \ , \tag{27}$$

where $D_{ba}(g)$ is a representation of the same group element g as $U(g)$ on a different space. Using Zwi (33), i.e. that D_{ba}s are orthogonal matrices ($D^{-1}_{ba} = D_{ab}$):

$$J^n u = \left(g\bar\psi\gamma^\nu T^r_a\psi\right)T^r_a \to \left(g\bar\psi U^\dagger\gamma^\nu T^r_a U\psi\right)T^r_b = \left(g\bar\psi\gamma^\nu T^r_b D^{-1}_{ba}\psi\right)T^r_a = J^{b\nu}T^r_a D_{ab} = J^{a\nu}\ T^r_b D_{ba} \ , \tag{28}$$

which is the same transformation law, as in (27). Hence the equation of motion transforms covariantly.

Secondly, we do a direct calculation for $SU(N)$. Putting matrix indices $i, j, \ldots$ on

$$J^\nu_{ij} = J^{nua}(T^a)_{ij} = g\bar\psi_l\gamma^\nu(T^a)_{lm}\psi_l(T^a)_{ij} \tag{29}$$

$$= \frac{g}{2}(\bar\psi_j\gamma^\nu\psi_i - \frac{1}{N}\delta_{ij}\bar\psi_l\gamma^\nu\psi_l) \tag{30}$$

In the third line, we have used the identity for $SU(N)$

$$(T^a)_{lm}(T^a)_{ij} = \frac{1}{2}(\delta_{lj}\delta_{mi} - \frac{1}{N}\delta_{lm}\delta_{ij}) \tag{31}$$

which can be found, eg., as equation (A.38) of Peskin. Thus under a gauge transformation $\psi \to U\psi$ and

$$J^\nu_{ij} \to \frac{g}{2}(\bar\psi_l U^\dagger_{lj}\gamma^\nu U_{im}\psi_m - \frac{1}{N}\delta_{ij}\bar\psi_k U^\dagger_{kl}\gamma^\nu U_{lm}\psi_m) = U_{im}\frac{g}{2}(\bar\psi_l\gamma^\nu\psi_m - \frac{1}{N}\delta_{lm}\bar\psi_k\gamma^\nu\psi_k)U^\dagger_{lj} = \left(UJ^\nu U^\dagger\right)_{ij} \tag{32}$$

i.e. J^ν transforms in the adjoint.

3. (a) We start with the case of QED. Because $\mathcal{L} = A^\mu J_\mu + \ldots$, and the current is odd under C, A_μ is also odd under C. This should hold for the non-abelian gauge field A_μ, but not necessarily for its components A^a_μ! (Note that $CT^aC = -T^a$.) ∂_μ has no transformation under C so F is odd, but $\mathcal{L}_\theta$ is even.

   Under P, A_t and ∂_t are even, whereas ∂_i and A_i are odd. Since $\mathcal{L}_\theta$ has an epsilon tensor, each index ($\{t, 1, 2, 3\}$) must occur exactly once. Since the spatial directions are odd, we get that $\mathcal{L}_\theta$ is odd under parity.

   The discussion for T is equivalent, except with the roles of space and time exchanged, and so $\mathcal{L}_\theta$ is odd under T. We note with satisfaction that $\mathcal{L}_\theta$ is still even under CPT. (However, the discussion for components is again more complicated in this case. Because T is anti-unitary to keep the commutation relations in the group algebra invariant we need $TT^aT = -T^a$.)

   In summary $\mathcal{L}_\theta$ breaks P, T (and hence CP) and conserves C.

   (b) We are aiming at:

$$\mathcal{L}_\theta = \frac{\theta}{64\pi^2}\partial_\mu K^\mu \tag{33}$$

   Let us start with the most naive guess for K^μ:

$$\begin{aligned}2\partial_\mu\left[\epsilon^{\mu\nu\lambda\rho}A^a_\nu F^a_{\lambda\rho}\right] &= 2\epsilon^{\mu\nu\lambda\rho}\left(\left(\partial_\mu A^a_\nu\right)F^a_{\lambda\rho} + A^a_\nu\partial_\mu F^a_{\lambda\rho}\right)\\ &= \epsilon^{\mu\nu\lambda\rho}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)F^a_{\lambda\rho} + 2\epsilon^{\mu\nu\lambda\rho}A^a_\nu\left[\left(\partial_\mu\partial_\lambda A^a_\rho - \partial_\mu\partial_\rho A^a_\lambda\right) + gf^{abc}\left(\left(\partial_\mu A^b_\lambda\right)A^c_\rho + A^b_\lambda\partial_\mu A^c_\rho\right)\right]\end{aligned} \tag{34}$$

   The first term can be completed to give $F\widetilde{F}$:

$$\epsilon^{\mu\nu\lambda\rho}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)F^a_{\lambda\rho} = \epsilon^{\mu\nu\lambda\rho}F^a_{\mu\nu}F^a_{\lambda\rho} - gf^{abc}A^b_\mu A^c_\nu F^a_{\lambda\rho} \tag{35}$$

Plugging this into (34) – after relabeling indices – we obtain:

$$2\partial_\mu\left[\epsilon^{\mu\nu\lambda\rho}A^a_\nu F^a_{\lambda\rho}\right]=\epsilon^{\mu\nu\lambda\rho}F^a_{\mu\nu}F^a_{\lambda\rho}+2\epsilon^{\mu\nu\lambda\rho}gf^{abc}A^a_\mu A^b_\nu\partial_\lambda A^c_\rho \tag{36}$$

where we used $\epsilon^{\mu\nu\alpha\beta}A^e_\mu A^d_\alpha A^b_\beta A^c_\nu f^{abc}f^{aed}=0$, which is true by the Jacobi identity. (This is easier to see in the matrix notation: $\epsilon^{\mu\nu\lambda\rho}\,\mathrm{tr}\left\{A_\mu\left[A_\nu,\left[A_\lambda,A_\rho\right]\right]\right\}=0$.) Our naive guess didn't turn out to be perfect, but notice that the surplus term has one less derivative then the RHS. Now it is easy to see that

$$2\epsilon^{\mu\nu\lambda\rho}gf^{abc}A^a_\mu A^b_\nu\partial_\lambda A^c_\rho=\frac{2}{3}\partial_\mu\left[\epsilon^{\mu\nu\lambda\rho}gf^{abc}A^a_\nu A^b_\lambda A^c_\rho\right] \tag{37}$$

and hence

$$\begin{aligned}K^\mu&=2\epsilon^{\mu\nu\lambda\rho}\left[A^a_\nu F^a_{\lambda\rho}-\frac{1}{3}gf^{abc}A^a_\nu A^b_\lambda A^c_\rho\right]=4\epsilon^{\mu\nu\lambda\rho}A^a_\nu\left[\partial_\lambda A^a_\rho+\frac{1}{3}gf^{abc}A^b_\lambda A^c_\rho\right]\\&=8\epsilon^{\mu\nu\lambda\rho}\,\mathrm{tr}\left[A_\nu\left(\partial_\lambda A_\rho-\frac{2ig}{3}A_\lambda A_\rho\right)\right]=4\epsilon^{\mu\nu\lambda\rho}\,\mathrm{tr}\left[A_\nu\left(F_{\lambda\rho}+\frac{2ig}{3}A_\lambda A_\rho\right)\right]\end{aligned} \tag{38}$$

where we listed some possible ways of writing the result. Note that K^μ does not have nice gauge transformation properties. It is a topological current used to characterize topological sectors in gauge theories.

4. (a) Under a gauge transformation

$$A_\mu\ \rightarrow\ UA_\mu U^\dagger-\frac{i}{g}\partial_\mu UU^\dagger\ . \tag{39}$$

To gauge away the field we thus need to solve for a U that satisfies

$$\partial_\mu U(x)+igU(x)A_\mu(x)=0\,. \tag{40}$$

Differentiating (40) with respect to x^ν we find

$$\begin{aligned}\partial_\nu\partial_\mu U+igU\partial_\nu A_\mu+ig(\partial_\nu U)A_\mu&=0\\\partial_\nu\partial_\mu U+igU\partial_\nu A_\mu+g^2UA_\nu A_\mu&=0\,,\end{aligned} \tag{41}$$

where we plugged in (40) to get to the second line. Antisymmetrizing in μ and ν we get

$$0=(\partial_\nu\partial_\mu-\partial_\nu\partial_\mu)U(x)=-igU(x)\,F_{\nu\mu} \tag{42}$$

Thus $F_{\mu\nu}=0$ implies $[\partial_\mu,\partial_\nu]\,U=0$. (If $F_{\mu\nu}\neq0$ the field configuration is not equivalent to zero, and (40) cannot be solved.)

(b) Equation (40) must hold at any point in spacetime and differentiating along any direction. In particular, given a path parametrised by $x^\mu(\bar{s}),\ \bar{s}\in[0,s],\ x(0)=x_0$ we contract (40) with the tangent vector at each point on the curve, to obtain differential equation for $U(x(\bar{s}))$:

$$\frac{dx^\mu}{d\bar{s}}\frac{\partial U}{\partial x^\mu}=\frac{dU(x(\bar{s}))}{d\bar{s}}=-\frac{dx^\mu}{d\bar{s}}igU(x(\bar{s}))A_\mu(x(\bar{s}))\ , \tag{43}$$

with the initial condition $U(x(0))=1$. This equation should be familiar, it is a Schrödinger-like equation that is satisfied by the non-abelian Wilson line (see for example PS equation (15.57)). The solution is

$$U^\dagger(x(s))=V(x(s))=P\big\{\exp\big(-ig\int_0^s d\bar{s}\,\frac{dx^\mu}{d\bar{s}}A_\mu(x(\bar{s}))\big)\big\}\ , \tag{44}$$

where P is the path-ordering operator.

Solving (43) is not equivalent to solving (40) because (43) only ensures that the projection of the left hand side of (40) along the tangent of the path is zero. However, if the solution of (43) for $U(x(s))$ is independent of the path, then we can choose a path with a different tangent direction at $x(s)$ and conclude that the component of the left hand side of (40) is zero along the other direction too. In four dimensions we need four paths starting from x_0 with linearly independent tangent vectors to demonstrate that (40) is satisfied. So, a path-independent solution $U(x(s))$ of (43) implies that the solution of (43) along any path provides a solution of (40).

(c) Consider the change in $U(s)$ when the path is changed from $x^\mu(\bar{s})$ to $x^\mu(\bar{s}) + \delta x^\mu(\bar{s})$ with fixed endpoints: $\delta x^\mu(0) = 0$ and $\delta x^\mu(s) = 0$. Writing $U(\bar{s}) \to U(\bar{s}) + \delta U(\bar{s})$. We find that the change in equation (43) is

$$\frac{d\delta U}{d\bar{s}} + ig\frac{dx^\mu}{d\bar{s}}\delta U A_\mu + igU\left(\frac{\partial A_\mu}{\partial x^\nu}\delta x^\nu \frac{dx^\mu}{d\bar{s}} + A_\mu \frac{d\delta x^\mu}{d\bar{s}}\right) = 0\,. \tag{45}$$

We can now consider $\frac{d}{ds}(\delta U U^\dagger)$ [1] . Using the hermitian conjugate of (43), we find

$$\begin{aligned}\frac{d}{d\bar{s}}(\delta U U^\dagger) &= \delta U\ \frac{dU^\dagger}{d\bar{s}} + \frac{d\delta U}{d\bar{s}}\ U^\dagger = -ig\,U\left(\frac{\partial A_\mu}{\partial x^\nu}\frac{dx^\mu}{d\bar{s}}\delta x^\nu + A_\mu\frac{d\delta x^\mu}{d\bar{s}}\right)U^\dagger \\ &= -ig\,U\frac{\partial A_\mu}{\partial x^\nu}\frac{dx^\mu}{d\bar{s}}U^\dagger\delta x^\nu + ig\frac{d}{d\bar{s}}(UA_\mu U^\dagger)\delta x^\mu - ig\frac{d}{d\bar{s}}(UA_\mu U^\dagger \delta x^\mu)\,.\end{aligned} \tag{46}$$

Integrating (46) over the path from $\bar{s} = 0$ to $\bar{s} = s$, we obtain

$$\delta U(s)U^\dagger(s) - \delta U(0)U^\dagger(0) = -ig\int_0^s d\bar{s}\left[U\frac{\partial A_\mu}{\partial x^\nu}\frac{dx^\mu}{d\bar{s}}\delta x^\nu U^\dagger - \frac{d}{d\bar{s}}(UA_\nu U^\dagger)\delta x^\nu\right] - igU(\bar{s})A_\mu(\bar{s})U^\dagger(\bar{s})\delta x^\mu(\bar{s})\Big|_0^s \tag{47}$$

Since $\delta U(0) = 0$, $\delta x^\mu(0) = 0$, and $\delta x^\mu(s) = 0$,

$$\delta U(s)U^\dagger(s) = -ig\int_0^s d\bar{s}\left[U\frac{\partial A_\mu}{\partial x^\nu}\frac{dx^\mu}{d\bar{s}}\delta x^\nu U^\dagger - \frac{d}{d\bar{s}}(UA_\nu U^\dagger)\delta x^\nu\right] \tag{48}$$

The second term in the integrand is simplified using equation (43):

$$\frac{d}{ds}(UA_\nu U^\dagger) = \left(-igUA_\mu\frac{dx^\mu}{d\bar{s}}\right)A_\nu U^\dagger + U\left(\frac{\partial A_\nu}{\partial x^\mu}\frac{dx^\mu}{d\bar{s}}\right)U^\dagger + UA_\nu\left(igA_\mu\frac{dx^\mu}{d\bar{s}}U^\dagger\right)\,. \tag{49}$$

Plugging into (48) we get:

$$\delta U(s)U^\dagger(s) = -ig\int_0^s d\bar{s}\ \frac{dx^\mu}{d\bar{s}}\delta x^\nu U\left(\partial_\nu A_\mu - \partial_\mu A_\nu - ig\left[A_\nu, A_\mu\right]\right)U^\dagger = ig\int_0^s d\bar{s}\ \frac{dx^\mu}{d\bar{s}}\delta x^\nu U F_{\mu\nu}U^\dagger \tag{50}$$

If $F_{\mu\nu} = 0$ then $\delta U(s)U^\dagger(s) = 0$, so $\delta U(s) = 0$. We conclude that the solution is path independent for a flat connection.

5. (a) Let us assume that A_μ and $\widetilde{A}_\mu$ give the same field strength tensor:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \partial_\mu \widetilde{A}_\nu - \partial_\nu \widetilde{A}_\mu \tag{51}$$

$$a_\mu \equiv A_\mu - \widetilde{A}_\mu \tag{52}$$

$$0 = \partial_\mu a_\nu - \partial_\nu a_\mu = f_{\mu\nu} \tag{53}$$

Hence a_μ is an abelian gauge field with $f_{\mu\nu} = 0$, According to Problem 4. this implies that a_μ is gauge equivalent to zero. Hence there exists such an α:

[1] Alternatively we can consider $U^\dagger \delta U$.

$$0 = a_\mu + \frac{1}{e}\partial_\mu \alpha \tag{54}$$

$$-\frac{1}{e}\partial_\mu \alpha = a_\mu = A_\mu - \widetilde{A}_\mu \tag{55}$$

$$\widetilde{A}_\mu = A_\mu + \frac{1}{e}\partial_\mu \alpha \ , \tag{56}$$

which is the statement that A_μ and $\widetilde{A}_\mu$ are gauge equivalent locally.

(b) Recall

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right] \tag{57}$$

Suppose

$$A_x = -\frac{1}{2}gy\frac{\sigma^3}{2} \text{ and } A_y = \frac{1}{2}gx\frac{\sigma^3}{2} \tag{58}$$

then the field strength derives only from the derivative terms and the only nonzero components are

$$F_{xy} = g\frac{\sigma^3}{2} = -F_{yx} \tag{59}$$

Note that this field configuration is essentially Abelian, i.e. the fields only take values from a $U(1)$ subgroup of $SU(2)$.
Now suppose

$$A'_x = \frac{\sigma^1}{2} \text{ and } A'_y = \frac{\sigma^2}{2} \tag{60}$$

The field strength now derives entirely from the commutator terms and the only nonzero components are

$$F'_{xy} = g\frac{\sigma^3}{2} = -F'_{yx} \tag{61}$$

Hence the two field tensors are equal.

(c) Since the field strength has no coordinate dependence

$$D_\rho F_{\mu\nu} = \partial_\rho F_{\mu\nu} - ig\left[A_\rho, F_{\mu\nu}\right] \tag{62}$$

will come only from the commutator terms. For the connection A this will be zero, as both A and F are in the σ^3 direction

$$D_\rho F_{\mu\nu} = 0 \tag{63}$$

However, the commutator terms are non-zero for A'. One finds

$$D_x F'_{xy} = -g^2\frac{\sigma^2}{2} \text{ and } D_y F'_{xy} = g^2\frac{\sigma^1}{2} \tag{64}$$

If there were a gauge transformation between A and A' then this gauge transformation would also take $(D_x F_{xy})'$ to $D_x F_{xy}$. However, this is not possible, as the gauge transformation $U\left(D_x F_{xy}\right)U^\dagger$ cannot map zero to a non-zero object, because U is an invertible matrix. This is in contrast with the gauge transformation properties of a gauge field, which involves an inhomogenuous term.

6. (a) We can follow exactly the same steps we followed for the Lorentz gauge calculation. We impose a generalized axial gauge by inserting into the path integral a delta function

$$\delta(A_3^a(x) - \omega^a(x)) \tag{65}$$

where ω is some arbitrary function. We can then do a gaussian weighted integral over all

$$\int \mathcal{D}\omega^a \, exp\left[-\frac{1}{2\xi}\int (w^a)^2\right] \tag{66}$$

This adds a gauge fixing term to the lagrangian such that we have

$$\mathcal{L} = -\frac{1}{4}(F^a_{\mu\nu})^2 - \frac{1}{2\xi}(A^a_3)^2 + \ldots \tag{67}$$

In momentum space, the quadratic part of this Lagrangian is

$$\frac{1}{2}A^a_\sigma \left[-k^2\eta^{\sigma\mu} + k^\sigma k^\mu - \frac{1}{\xi}\delta^\sigma_3\delta^\mu_3\right]\delta^{ab}A^b_\mu \tag{68}$$

This operator can be inverted. It can be checked that the inverse is

$$-\frac{i}{k^2 - i\epsilon}\left[\eta_{\mu\nu} + \frac{(\xi k^2 + 1)k_\mu k_\nu}{k_3^2} - \frac{k_\mu k_3 \delta^3_\nu + k_\nu k_3 \delta^3_\mu}{k_3^2}\right]\delta^{ab} \tag{69}$$

For $\xi \to 0$, the integral (66) oscillates very quickly except around $\omega^a = 0$ thus imposing the axial gauge $A^a_3 = 0$. In this limit the propagator becomes

$$-\frac{i}{k^2 - i\epsilon}\left[\eta_{\mu\nu} + \frac{k_\mu k_\nu}{k_3^2} - \frac{k_\mu k_3 \delta^3_\nu + k_\nu k_3 \delta^3_\mu}{k_3^2}\right]\delta^{ab} \tag{70}$$

Alternatively, one need not introduce the fixing function ω at all. Setting $A_3 = 0$ in the original Lagrangian gives a quadratic part in momentum space that is

$$\frac{1}{2}A^a_{\hat\sigma}\left[-k^2\eta^{\hat\sigma\hat\mu} + k^{\hat\sigma}k^{\hat\mu}\right]\delta^{ab}A^b_{\hat\mu} \tag{71}$$

where hatted indices run over $0, 1, 2$ but k^2 includes all momentum components. This can be inverted to give a propagator

$$-\frac{1}{k^2 - i\epsilon}\left[\eta_{\hat\mu\hat\nu} + \frac{k_{\hat\mu}k_{\hat\nu}}{k_3^2}\right]\delta^{ab} \tag{72}$$

These two methods are equivalent as this propagator is just the non-zero 3×3 block of (70).

(b) When $f_a = \partial_i A^i_a$. things are very similar to the Lorentz gauge, except now indices run over only spatial indices. The ghost Lagrangian becomes

$$\mathcal{L}_{ghost} = \bar{c}^a(\partial_i\partial^i\delta^{ac} - gf^{abc}\partial^i A^b_i)c^b \tag{73}$$

The propagator in momentum space is

$$\frac{i}{\vec{k}^2}\delta^{ab} \tag{74}$$

where $\vec{k}^2$ only involves spatial components.

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