Problem Set 4 Solutions

1. (a) We are going to use Peskin's conventions in order not to conflict with the instructions in the problem. The interaction vertex for this theory is $-ig\gamma^\mu$. The amplitude for the e^+e^- annihilation into B is:

$$i\mathcal{M} = \bar{v}(p_+)(-ig\gamma^\mu)u(p_-)\ \epsilon^*_\mu(q)\ , \tag{1}$$

where $p_\pm$ is the momentum of $e^\pm$ and q is the momentum of B. We average over initial polarizations of the fermions and some over the final state polarizations of B to get:

$$\begin{aligned}
\overline{|\mathcal{M}|^2} &\equiv \frac{1}{4}\sum_{s_+,s_-,i}|\mathcal{M}|^2 = \frac{1}{4}\sum_{s_+,s_-,i}\mathcal{M}^*\mathcal{M} \\
&= \frac{g^2}{4}\sum_{s_+,s_-,i}\mathrm{tr}\left[\left(\bar{u}_{s_-}(p_-)\gamma^\nu v_{s_+}(p_+)\ \epsilon^{(i)}_\nu(q)\right)\left(\bar{v}_{s_+}(p_+)\gamma^\mu u_{s_-}(p_-)\ \epsilon^{(i)*}_\mu(q)\right)\right] \\
&= \frac{g^2}{4}(-\eta_{\mu\nu})\sum_{s_+,s_-}\mathrm{tr}\left[\gamma^\nu v_{s_+}(p_+)\bar{v}_{s_+}(p_+)\gamma^\mu u_{s_-}(p_-)\bar{u}_{s_-}(p_-)\right] \\
&= -\frac{g^2}{4}\mathrm{tr}\left[\gamma^\mu \not{p}_+\gamma_\mu \not{p}_-\right] = -\frac{g^2}{4}\mathrm{tr}\left[(-2\not{p}_+)\not{p}_-\right] = 2g^2 p_+\cdot p_- \\
&= g^2q^2 = g^2M^2
\end{aligned} \tag{2}$$

where in the third line we used the replacement $\sum_i \epsilon^{(i)}_\nu\epsilon^{(i)*}_\mu \to (-\eta_{\mu\nu})$, in the fourth we neglected the mass of the electron and used well-known properties of spinor and also some identities for Dirac matrices, finally in the fith we used the kinematics of the problem. Now the cross section is easy to calculate in the CM frame:

$$\begin{aligned}
\sigma(e^+e^-\to B) &= \frac{1}{8E^2}\int\frac{d^3q}{2E_q(2\pi)^3}\ (2\pi)^4\delta^{(4)}(p_++p_--q)\ \overline{|\mathcal{M}|^2} \\
&= \frac{1}{2s}\frac{\pi}{M}\ \delta(2E-M)\ g^2M^2 = \pi g^2\ \delta(s-M^2)
\end{aligned} \tag{3}$$

The decay rate is calculated by reusing the amplitude of our previous calculation. Here however, we average over B polarizations to get a factor of 1/3:

$$\Gamma(B\to e^+e^-) = \frac{1}{2M}\int d\Pi_2\ \left(\frac{1}{3}\,|\mathcal{M}|^2\right) = \frac{1}{2M}\left(\frac{4g^2M^2}{3}\right)\int d\Pi_2 = \frac{g^2M}{12\pi} \tag{4}$$

We conclude that

$$\sigma(e^+e^-\to B) = \frac{12\pi^2}{M}\Gamma(B\to e^+e^-)\delta(s-M^2)\ , \tag{5}$$

in agreement with PS Sec. 5.3, where a vector meson bound state was considered.

(b) The Feynman graphs corresponding to the process are the same as the ones on page 168 in PS. Their contribution is:

$$i\mathcal{M} = \bar{v}(p_+)\left[(-ig\gamma^\mu)\epsilon^*_\mu(q_B)\frac{1}{-i(\not{p}_- - \not{q}_\gamma)}(-ie\gamma^\nu)\epsilon^*_\nu(q_\gamma) + (-ig\gamma^\mu)\epsilon^*_\mu(q_\gamma)\frac{1}{i(\not{p}_+ - \not{q}_\gamma)}(-ie\gamma^\nu)\epsilon^*_\nu(q_B)\right]u(p_-)\ , \tag{6}$$

which after averaging over $s_\pm$ and summing over γ and B polarizations becomes:

$$\begin{aligned}
\overline{|\mathcal{M}|^2} &= \frac{1}{4}\sum_{s_+,s_-,i_\gamma,i_B} \mathcal{M}^*\mathcal{M} \\
&= \frac{g^2e^2}{4}\,\mathrm{tr}\left[\not{p}_-\left(\gamma^\mu\frac{\not{p}_- - \not{q}_\gamma}{(p_- - q_\gamma)^2}\gamma^\nu - \gamma^\nu\frac{\not{p}_+ - \not{q}_\gamma}{(p_+ - q_\gamma)^2}\gamma^\mu\right)\not{p}_+\left(\gamma_\nu\frac{\not{p}_- - \not{q}_\gamma}{(p_- - q_\gamma)^2}\gamma_\mu - \gamma_\mu\frac{\not{p}_+ - \not{q}_\gamma}{(p_+ - q_\gamma)^2}\gamma_\nu\right)\right] \\
&= \frac{g^2e^2}{4}\,\mathrm{tr}\left[\frac{1}{(2p_-\cdot q_\gamma)^2}\not{p}_-\gamma^\mu(\not{p}_- - \not{q}_\gamma)\gamma^\nu\not{p}_+\gamma_\nu(\not{p}_- - \not{q}_\gamma)\gamma_\mu\right. \\
&\quad -\frac{1}{(2p_-\cdot q_\gamma)(2p_+\cdot q_\gamma)}\not{p}_-\gamma^\mu(\not{p}_- - \not{q}_\gamma)\gamma^\nu\not{p}_+\gamma_\mu(\not{p}_+ - \not{q}_\gamma)\gamma_\nu \\
&\quad -\frac{1}{(2p_-\cdot q_\gamma)(2p_+\cdot q_\gamma)}\not{p}_-\gamma^\nu(\not{p}_+ - \not{q}_\gamma)\gamma^\mu\not{p}_+\gamma_\nu(\not{p}_- - \not{q}_\gamma)\gamma_\mu \\
&\quad \left. +\frac{1}{(2p_+\cdot q_\gamma)^2}\not{p}_-\gamma^\nu(\not{p}_+ - \not{q}_\gamma)\gamma^\mu\not{p}_+\gamma_\mu(\not{p}_+ - \not{q}_\gamma)\gamma_\nu\right]
\end{aligned} \tag{7}$$

It is a bit tedious to deal with all the traces, but using the following identities and the cyclic property of the trace they can be calculated:

$$\gamma^\mu\not{p}\gamma_\mu = -2\not{p} \tag{8}$$

$$\gamma^\mu\not{p}_1\not{p}_2\gamma_\mu = 4(p_1\cdot p_2) \tag{9}$$

$$\gamma^\mu\not{p}_1\gamma^\nu\not{p}_2\gamma_\mu = -2\not{p}_2\gamma^\nu\not{p}_1 \tag{10}$$

So one by one the traces are:

$$\mathrm{tr}\left[\not{p}_-\gamma^\mu(\not{p}_- - \not{q}_\gamma)\gamma^\nu\not{p}_+\gamma_\nu(\not{p}_- - \not{q}_\gamma)\gamma_\mu\right] = 4\,\mathrm{tr}\left[\not{p}_-(\not{p}_- - \not{q}_\gamma)\not{p}_+(\not{p}_- - \not{q}_\gamma)\right] = 32(p_-\cdot q_\gamma)(p_+\cdot q_\gamma) \tag{11}$$

$$\begin{aligned}
\mathrm{tr}\left[\not{p}_-\gamma^\mu(\not{p}_- - \not{q}_\gamma)\gamma^\nu\not{p}_+\gamma_\mu(\not{p}_+ - \not{q}_\gamma)\gamma_\nu\right] &= -2\,\mathrm{tr}\left[\not{p}_-\not{p}_+\gamma^\nu(\not{p}_- - \not{q}_\gamma)(\not{p}_+ - \not{q}_\gamma)\gamma_\nu\right] \\
&= -8\left[(p_- - q_\gamma)\cdot(p_+ - q_\gamma)\right]\mathrm{tr}\left[\not{p}_-\not{p}_+\right] \\
&= -32\left[(p_- - q_\gamma)\cdot(p_+ - q_\gamma)\right](p_-\cdot p_+) \\
&= -32\,\frac{s+t+u}{2}\,(p_-\cdot p_+) = -16M^2(p_-\cdot p_+)
\end{aligned} \tag{12}$$

$$\mathrm{tr}\left[\not{p}_-\gamma^\nu(\not{p}_+ - \not{q}_\gamma)\gamma^\mu\not{p}_+\gamma_\nu(\not{p}_- - \not{q}_\gamma)\gamma_\mu\right] = -16M^2(p_-\cdot p_+) \tag{13}$$

$$\mathrm{tr}\left[\not{p}_-\gamma^\nu(\not{p}_+ - \not{q}_\gamma)\gamma^\mu\not{p}_+\gamma_\mu(\not{p}_+ - \not{q}_\gamma)\gamma_\nu\right] = 4\,\mathrm{tr}\left[\not{p}_-(\not{p}_+ - \not{q}_\gamma)\not{p}_+(\not{p}_+ - \not{q}_\gamma)\right] = 32(p_-\cdot q_\gamma)(p_+\cdot q_\gamma) \tag{14}$$

To sum it all up we get

$$\overline{|\mathcal{M}|^2} = 2g^2e^2\left[\frac{p_+\cdot q_\gamma}{p_-\cdot q_\gamma} + \frac{M^2(p_-\cdot p_+)}{(p_-\cdot q_\gamma)(p_+\cdot q_\gamma)} + \frac{p_-\cdot q_\gamma}{p_+\cdot q_\gamma}\right] = 2g^2e^2\left[\frac{u}{t} + \frac{2M^2s}{tu} + \frac{t}{u}\right] \tag{15}$$

In the CM frame the explicit expressions for the momenta are:

$$p_- = (E,0,0,E) \qquad p_+ = (E,0,0,-E) \tag{16}$$

$$q_\gamma = (E_\gamma, E_\gamma\sin\Theta, 0, E_\gamma\cos\Theta) \qquad q_B = (E_B, -E_\gamma\sin\Theta, 0, -E_\gamma\cos\Theta) \tag{17}$$

$$2E = E_\gamma + E_B \qquad M^2 = E_B^2 - E_\gamma^2 \implies E_\gamma = \frac{4E^2 - M^2}{4E} \qquad E_B = \frac{4E^2+M^2}{4E} \tag{18}$$

With these relations the Mandelstam variables are:

$$s = 2p_-\cdot p_+ = 4E^2 \tag{19}$$

$$t = -2p_-\cdot q_\gamma = -2EE_\gamma(1-\cos\Theta) \tag{20}$$

$$u = -2p_+\cdot q_\gamma = -2EE_\gamma(1+\cos\Theta) \tag{21}$$

Finally

$$\overline{|\mathcal{M}|^2} = 2g^2e^2\,\frac{t^2+u^2+2M^2s}{tu} = 4g^2e^2\,\frac{1+\cos^2\Theta + M^2/E_\gamma^2}{1-\cos^2\Theta} \tag{22}$$

$$\frac{d\sigma}{d\cos\Theta} = \frac{1}{8E^2}\frac{E_\gamma}{16\pi E}\,\overline{|\mathcal{M}|^2} = \frac{g^2e^2E_\gamma}{32\pi E^3}\,\frac{1+\cos^2\Theta+M^2/E_\gamma^2}{1-\cos^2\Theta}\ , \tag{23}$$

which reproduces PS (5.106) for $m_e = 0$ and $M << E_\gamma$, which is the high energy limit.

(c) Notice that (23) diverges for $\Theta \to 0$ or π. This divergence is cutoff by the electron mass which appears in the denominator next to the momentum invariant that gives rise to a collinear singularity. With the mass of the electron kept to the first non-trivial order and by (non-systematically) expanding around $\Theta = 0$ we get:

$$p_- = (E, 0, 0, \sqrt{E^2 - m^2}) = E\left(1, 0, 0, \left(1 - \frac{m^2}{2E^2}\right)\right) + \mathcal{O}\left(\frac{m^4}{E^4}\right) \tag{24}$$

$$q_\gamma = E_\gamma(1, \sin\Theta, 0, \cos\Theta) = xp_- + \mathcal{O}(\Theta) \qquad x \equiv \frac{E_\gamma}{E} \tag{25}$$

$$p'_- \equiv p_- - q_\gamma \approx (1 - x)p_- + \mathcal{O}(\Theta) \tag{26}$$

$$t = (q_\gamma - p_-)^2 = -2q_\gamma \cdot p_- + m^2 = -2xE^2\left(1 - \cos\Theta + \frac{m^2}{2E^2}\right) + \mathcal{O}(m^2\Theta^2) \tag{27}$$

$$M^2 = (p'_- + p_+)^2 = ((1 - x)p_- + \mathcal{O}(\Theta) + p_+)^2 = (1 - x)s + \mathcal{O}(E\Theta) \tag{28}$$

With these momentum invariants we get near $\Theta = 0$:

$$\frac{1}{t} \propto \frac{1}{1 - \cos\Theta} \to \frac{1}{1 - \cos\Theta + \frac{m^2}{2E^2}} \tag{29}$$

$$\frac{d\sigma}{d\cos\Theta} \to \frac{g^2e^2E_\gamma}{32\pi E^3} \frac{1 + \cos^2\Theta + \frac{M^2}{x^2E^2}}{(1 + \cos\Theta)\left(1 - \cos\Theta + \frac{m^2}{2E^2}\right)} \approx \frac{g^2e^2x}{32\pi E^2} \frac{2 + \frac{M^2}{x^2E^2}}{2\left(1 - \cos\Theta + \frac{m^2}{2E^2}\right)} \tag{30}$$

$$\begin{aligned}\sigma(e^+e^- \to \gamma(\text{forward})B) &= \int_0 d\cos\Theta \frac{d\sigma}{d\cos\Theta} = \frac{g^2e^2}{32\pi} \frac{x + \frac{(1-x)s}{2xE^2}}{E^2} \int_0 d\cos\Theta \frac{1}{\left(1 - \cos\Theta + \frac{m^2}{2E^2}\right)} \\ &= \frac{g^2e^2}{4\pi} \frac{\frac{x^2}{2} + (1 - x)}{xs} \log\left(\frac{2E^2}{m^2}\right) \approx \frac{g^2e^2}{4\pi} \frac{\frac{x^2}{2} + (1 - x)}{xs} \log\left(\frac{s}{m^2}\right) ,\end{aligned} \tag{31}$$

where we used that $s = 4E^2$ and that we are only interested in leading terms in s/m^2. Now it is just a matter of algebra to check that:

$$f(x) = \frac{\alpha}{2\pi} \frac{1 + (1 - x)^2}{x} \log\left(\frac{s}{m^2}\right) \tag{32}$$

$$\sigma(e^+e^- \to \gamma(\text{forward})B) \approx \int_0^1 dx\ f(x)\sigma(e^+e^- \to B)|_{E_{CM}^2 = (1-x)s} \tag{33}$$

$$= \int_0^1 dx\ \frac{\alpha}{2\pi} \frac{1 + (1 - x)^2}{x} \log\left(\frac{s}{m^2}\right) \pi g^2\ \delta\left((1 - x)s - M^2\right) \tag{34}$$

$$= \frac{e^2g^2}{8\pi} \log\left(\frac{s}{m^2}\right) \int_0^1 dx\ \frac{1 + (1 - x)^2}{x} \delta\left((1 - x)s - M^2\right) \tag{35}$$

$$= \frac{e^2g^2}{8\pi} \frac{1 + (1 - x)^2}{xs} \log\left(\frac{s}{m^2}\right) , \tag{36}$$

which is identical to (31) with $1 - x = M^2/s$ (see (28)). This formula after reinterpretation leads to the Gribov-Lipatov equations, which describe how the electron is seen to be a cloud of photons, electrons and positrons as we probe it with higher and higher energies. The QCD version of these ideas lead to the Altarelli-Parisi equations that are an essential tool for making predictions in hadronic processes.

2. (a) There are three diagrams which contribute: A tree level vertex, a one loop diagram, and the lowest order counter term.

   (b) The contribution from the three diagrams is

$$iV(k_1, k_2, k_3) = ig + igC + (ig)^3 \int \frac{d^dl}{(2\pi)^d} \frac{-i}{(l - k_1)^2 + m^2} \frac{-i}{l^2 + m^2} \frac{-i}{(l + k_2)^2 + m^2} \tag{37}$$

We can combine the three terms in the integral via the usual Feynman parameters, completing the square in the denominator, shifting the variable of integration, etc. We get

$$iV(k_1, k_2, k_3) = ig + igC + 2g^3 \int dx\, dy\, dz\, \delta(x + y + z - 1) \int \frac{d^dq}{(2\pi)^d} \frac{1}{(q^2 + \Delta)^3} \tag{38}$$

with

$$q \equiv l - (k_1 x - k_2 y) \tag{39}$$
$$\Delta \equiv k_1^2 zx + k_2^2 yz + k_3^2 xy + m^2 \tag{40}$$

Wick rotating, we get

$$V(k_1,k_2,k_3) = g + gC + 2g^3 \int dx\,dy\,dz\,\delta(x+y+z-1) \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2+\Delta)^3} \tag{41}$$

We can evaluate the momentum integral by dimensional regularization; we get

$$V(k_1,k_2,k_3) = g + gC + g^3 \int dx\,dy\,dz\,\delta(x+y+z-1) \frac{\Gamma(3-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}} \left(\frac{1}{\Delta}\right)^{3-\frac{d}{2}} \tag{42}$$

(c) For $d < 6$, the Gamma function is away from its poles and the integral is UV finite. For $d = 6 + n$ ($n \in \mathbb{N}$) we get a pole of the Gamma function, otherwise the integral is finite. This result agrees with the superficial degree of divergence (i.e. power counting) of our one loop Feynman graph.

(d) At $d = 6$ we first separate the dimensions of $g \to g\mu^{\epsilon/2}$ by introducing an arbitrary scale μ and then expand in ϵ ($d = 6 - \epsilon$), hence from (42):

$$\frac{V(k_1,k_2,k_3)}{g\mu^{\epsilon/2}} = 1 + C + \frac{g^2}{(4\pi)^3} \int dx\,dy\,dz\,\delta(x+y+z-1)\,\Gamma\left(\frac{\epsilon}{2}\right)\left(\frac{4\pi\mu^2}{\Delta}\right)^{\frac{\epsilon}{2}} \tag{43}$$
$$\Gamma\left(\frac{\epsilon}{2}\right) = \frac{2}{\epsilon} - \gamma + O(\epsilon) \tag{44}$$
$$\left(\frac{4\pi\mu^2}{\Delta}\right)^{\frac{\epsilon}{2}} = 1 + \frac{\epsilon}{2}\log\left(\frac{4\pi\mu^2}{\Delta}\right) + O(\epsilon^2) \tag{45}$$

We get as $\epsilon \to 0$

$$\frac{V(k_1,k_2,k_3)}{g\mu^{\epsilon/2}} = 1 + C + \alpha \int dx\,dy\,dz\,\delta(x+y+z-1)\left(\frac{2}{\epsilon} + \log\left(\frac{4\pi\mu^2}{\Delta}\right) - \gamma\right), \tag{46}$$

where $\alpha = g^2/(4\pi)^3$. Requiring $V(0,0,0) = g$ immediately gives

$$C = -\alpha \int dx\,dy\,dz\,\delta(x+y+z-1)\left(\frac{2}{\epsilon} + \log\left(\frac{4\pi\mu^2}{m^2}\right) - \gamma\right) = -\alpha\left(\frac{1}{\epsilon} + \frac{1}{2}\log\left(\frac{4\pi\mu^2 e^{-\gamma}}{m^2}\right)\right), \tag{47}$$

where we used that $\Delta(k_1 = k_2 = k_3 = 0) = m^2$. Plugging back in, we get the finite result

$$V(k_1,k_2,k_3) = g\left[1 - \alpha \int dx\,dy\,dz\,\delta(x+y+z-1)\,\log\frac{\Delta}{m^2}\right] + O(g^4) \tag{48}$$

(e) As $|k_3|^2 \gg m^2, \|k_1|^2, \|k_2|^2$ we have $\Delta/m^2 \to xyk_3^2/m^2$. Now

$$\int dx\,dy\,dz\,\delta(x+y+z-1)\log\left(\frac{xyk_3^2}{m^2}\right) = \int dx\,dy\,dz\,\delta(x+y+z-1)\left[\log xy + \log\left(\frac{k_3^2}{m^2}\right)\right] \tag{49}$$

The first term here is just some $O(1)$ number which is small compared to $\log\left(k_3^2/m^2\right)$. Integrating over the second term (which is x, y, z independent) just gives a factor of $1/2$. Thus, the result is

$$V(k_1,k_2,k_3) = g\left[1 - \frac{\alpha}{2}\left(\log\left(\frac{k_3^2}{m^2}\right) + O(1)\right)\right] + O(g^4) \tag{50}$$

which increases logarithmically with momentum. Notice that the loop correction can become $O(1)$ as $|k_3|^2$ grows big and the perturbative calculation is no more to be trusted. The RG method helps us in this situation by resumming large logarithms and absorbing them into the running coupling.