PROFESSOR: OK, let's start. So we are going to change perspective again and think in terms of lattice models. So for the first part of this course, I was trying to change your perspective from thinking in terms of microscopic degrees of freedom to a statistical field. Now we are going to go back and try to build pictures around things that look more microscopic.

Typically, in many solid state configurations, we are dealing with transitions that take place on a lattice. For example, imagine a square lattice, which is easy to draw, but there could be all kinds of cubic and more complex lattices. And then, at each side of this lattice, you may have one microscopic degrees of freedom that is ultimately participating in the ordering and the phase transition that you have in mind. Could, for example, be a spin, or it could be one atom in a binary mixture.

And what you would like to do is to construct a partition function, again, by summing over all degrees of freedom. And we need some kind of Hamiltonian. And what we are going to assume governs that Hamiltonian is the analog of this locality assumption that we have in statistical field theories. That is, we are going to assume that one of our degrees of freedom essentially talks to a small neighborhood around it. And the simplest neighborhood would be to basically talk to the nearest neighbor.

So if I, for example, assign index i to each side of the lattice, let's say I have some variable at each side that could be something. Let's call it SI. Then my partition function would be obtained by summing over all configurations. And the weight I'm going to assume in terms of this lattice picture to be a sum over interactions that exist between pairs of sides. So that's already an assumption that it's a pairwise thing. And I'm going to use this symbol ij with an angular bracket along it, around it, to indicate the sum over nearest neighbors. And there is some function of the

variables that live on these neighbors.

So basically, in the picture that I have drawn, interactions exist only between the places that you see lines. So that this pin does not interact with this pin, this pin, or this pin, but it interacts with these four pins, to which it is near neighbor. Now clearly, the form of the interaction has to be dictated by your degrees of freedom. And the idea of this representation, as opposed to the previous statistical field theory that we had, is that in several important instances, you may want to know not only what these universal properties are, but also, let's say, the explicit temperature or phase diagram of the system as a function of external parameters as well as temperature. And if you have some idea of how your microscopic degrees of freedom interact with each other, you should be able to solve this kind of partition function, get the singularities, et cetera.

So let's look at some simple versions of this construction and gradually discuss the kinds of things that we could do with it. So some simple models. The simplest one is the Ising model where the variable that you have at each side has two possibilities. So it's a binary variable in the context of a binary alloy. It could be, let's say, atom A or atom B that is occupying a particular site.

There are also cases where there will be some-- if this is a surface and you're absorbing particles on top of it, there could be a particle sitting here or not sitting here. So that would be also another example of a binary variable. So you could indicate that by empty or occupied, zero or one. But let's indicate it by plus or minus one as the two possible values that you can have.

Now, if I look at the analog of the interaction that I have between two sites that are neighboring each other, what can I write down? Well, the most general thing that I can write down is, first of all, a constant I can put, such as shift of energy. There could be a linear term in-- let's put all of these with minus signs. Sigma i and sigma j. I assume that it is symmetric with respect to the two sides. For reasons to become apparent shortly, I will divide by z, which is the coordination number. How many bonds per side? And then the next term that I can put is something like j sigma i

sigma j.

And actually, I can't put anything else. Because if you s of this as a power series in sigma i and sigma j, and sigma has only two values, it will terminate here. Because any higher power of sigma is either one or sigma itself. So another way of writing this is that the partition function of the Ising model is obtained by summing over all 2 to the n configurations that I have.

If I have a lattice of n sides, each side can have two possibilities of a kind of Hamiltonian, which is basically some constant plus h sum over i sigma i kind of field that prefers one side or the other side. So it is an analog of a magnetic field in this binary system. And basically, I convert it from a description that is overall balanced to a description overall sides. And that's why I had put the coordination number there. It's kind of a matter of convention.

And then a term that prefers neighboring sites to be aligned as long as k positive. So that's one example of a model. Another model that we will also look at is what I started to draw at the beginning. That is, at each site, you have a vector. And again, going in terms of the pictures that we had before, let's imagine that we have a vector that has n components. So SI is something that is in RN. And I will assume that the magnitude of this vector is one. So essentially, imagine that you have a unit vector, and each site that can rotate.

So if n equals to one, you have essentially one component vector. Its square has to be one, so the two values that it can have are plus one and minus one. So the n equals to one case is the Ising model again. So this ON is a generalization of the Ising model to multiple components. n equals to two corresponds to a unit vector that can take any angle in two dimensions. And that is usually given the name xy model. n equals to three is something that maybe you want to use to describe magnetic ions in this lattice. And classically, the direction of this ion could be anywhere. The spin of the ion can be anywhere on the surface of a sphere. So that's three components, and this model is sometimes called the Heisenberg model. Yes.

AUDIENCE: In the Ising model, what's the correspondence between g hat h hat j hat and g, h, and k?

PROFESSOR: Minus beta. Yeah. So maybe I should have written. In order to take this-- if I think of this as energy pair bond, then in order to get the Boltzmann weight, I have to put a minus beta. So I would have said that this g, for example, is minus beta g hat. k is minus beta j hat. And h is minus beta h hat.

So I should have emphasized that. What I meant by b-- actually, I wrote b-- so what I should have done here to be consistent, let's write this as minus beta. Thank you. That was important. If I described these b's as being energies, then minus beta times that will be what will go in the Boltzmann factor. OK

So whereas these were examples that we had more or less seen their continuum version in the Landau-Ginzburg model, there are other symmetries that get broken. And things for which we didn't discuss what the corresponding statistical field theory is. A commonly used case pertains to something that's called a Potts model. Where at each side you have a variable, let's call it SI, that takes q values, one, two, three, all the way to q.

And I can write what would appear in the exponent minus beta h to be a sum over nearest neighbors. And I can give some kind of an interaction parameter here, but a delta si sj. So basically, what it says is that on each side of a lattice, you put a number. Could be one, two, three, up to q. And if two neighbors are the same, they like it, and they gain some kind of a weight.

That is, if k is positive, encourages that to happen. If the two neighbors are different, then it doesn't matter. You don't gain energy, but you don't really care as to which one it is. The underlying symmetry that this has is permutation. Basically, if you were to permute all of these indices consistently across the lattice in any particular way, the energy would not change. So permutation symmetry is what underlies this.

And again, if I look at the case of two, then at each side, let's say I have one or two. And one one and two two are things that gain energy. One two or two one don't.

Clearly it is the same as the Ising model. So q equals to two is another way of writing the Ising model. Q equals to three is something that we haven't seen. So at each site, there are three possibilities.

Actually, when I started at MIT as a graduate student in 1979, the project that I had to do related to the q equals to three Potts model. Where did it come from? Well, at that time, people were looking at surface of graphite, which as you know, has this hexagonal structure. And then you can absorb molecules on top of that, such as, for example, krypton.

And krypton would want to come and sit in the center of one of these hexagons. But its size was such that once it sat there, you couldn't occupy any of these sides. So the next one would have to go, let's say, over here. Now, it is possible to subdivide this set of hexagons into three sub lattices. One, two, three, one, two, three, et cetera. Actually, I drew this one incorrectly. It would be sitting here. And what happens is that basically the agile particles would order by occupying one of three equivalent sublattices. So the way that that order got destroyed was then described by the q equals to three Potts universality class.

You can think of something like q equals to four that would have a symmetry of a tetragon. And so some structure that is like a tetragon getting, let's say, distorted in some particular direction would then have four equivalent directions, et cetera. So there's a whole set of other types of universality classes and symmetry breakings that we did not discuss before.

And I just want to emphasize that what we discussed before does not cover all possible symmetry breakings. It was just supposed to show you an important class and the technology to deal with that. But again, in this particular system, let's say you really wanted to know at what temperature the phase transition occurs, as well as what potential phase diagrams and critical behavior is.

And then you would say, well, even if I could construct a statistical field theory and analyze it in two dimensions, and we've seen how hard it is to go below some other critical dimension, it doesn't tell me things about phase diagrams, et cetera. So

maybe trying to understand and deal with this lattice model itself would tell us more information. Although about quantities that are not necessarily inverse.

Depending on your microscopic model, you may try to introduce more complicated systems, such as inspired by quantum mechanics, you can think of something that I'll call a spin S model in which your SI takes values from minus s, minus s plus 1, all the way to plus s. There's 2s plus 1 possibilities. And you can think of this as components of, say, a quantum spin of s along the zed axis. Write some kind of Hamiltonian for this. But as long as you deal with things classically, it turns out that this kind of system will not really have different universality from the Ising model.

So let's say we have this lattice model. Then what can we do? So in the next set of lectures, I will describe some tools for dealing with these models. One set of approaches, the one that we will start today, has to do with the position space renormalization group. That is the approach that we were following for renormalization previously. Dealt with going to Fourier space.

We had this sphere, hyper sphere. And then we were basically eliminating modes at the edge of this sphere in Fourier space. We started actually by describing the process in real space. So we will see that in some cases, it is possible to do a renormalization group directly on these lattice models. Second thing is, it turns out that as combinatorial problems, some, but a very small subset of these models, are susceptible to exact solutions.

Turns out that practically all models in one dimension, as we will start today, one can solve exactly. But there's one prominent case, which is the two dimensionalizing model that one can also solve exactly. And it's a very interesting solution that we will also examine in, I don't know, a couple of weeks. Finally, there are approximate schemes that people have evolved for studying these problems, where you have series expansions starting from limits, where you know what is happening.

And one simple example would be to go to very high temperatures. And at high temperatures, essentially every degree of freedom does what it wants. So it's essentially a zero dimensional problem that you can solve. And then you can start

treating interactions perturbatively. So this is kind of similar to the expansions that we had developed in 8 333, the virial expansions, et cetera, about the ideal gas limit. But now done on a system that is a lattice, and going to sufficiently high order that you can say something about the phase transition.

There is another extreme. In these systems, typically the zero temperature state is trivial. It is perfectly ordered. Let's say all the spins are aligned. And then you can start expanding in excitations around that state and see whether eventually, by including more and more excitations, you can see the phase transition out of the ordered state.

And something that is actually probably the most common use of these models, but I won't cover in class, is to put them on the computer and do some kind of a Monte Carlo simulation, which is essentially a numerical way of trying to generate configurations that are governed by this weight. And by changing the temperature as it appears in that weight, whether or not one can, in the simulation, see the phase transition happen.

So that's the change in perspective that I want you to have. So the first thing that we're going to do is to do the number one here, to do the position space renormalization group of one dimensional Ising model. And the procedure that I describe for you is sufficiently general that in fact you can apply to any other one dimensional model, as long as you only have these nearest neighbor interactions.

So here you have a lattice that is one dimensional. So you have a set of sites one, two. At some point, you have i, i minus one, i plus one. Let's say we call the last one n. So there are n sites. There are going to be 2 to the n possible configurations.

And your task is given that at each site, there's a variable that is binary. You want to calculate a partition function, which is a sum over all these 2 to the n configurations. Of a weight that is this e to the sum over i B, the interaction that couples SI and SI plus 1. Maybe I should have called this e hat. And B is the thing that has minus beta absorbed in it.

So notice that basically, the way that I have written it, one is interacting with two. i minus one is interacting with i. i is interacting with i plus one. So I wrote the nearest neighbor interaction in this particular fashion. We may or may not worry about the last spin, whether I want to finish the series here, or sometimes I will use periodic boundary condition and bring it back and couple it to the first one, where I have a ring. So that's another possibility. Doesn't really matter all that much at this stage.

So this runs for i going from one to n. There are n degrees of freedom. Now, renormalization group is a procedure by which I get rid of some degrees of freedom. So previously, I have emphasized that what we did was some kind of an averaging. So we said that let's say I could do some averaging of three sites and call some kind of a representative of those three.

Let's say that we want to do a RG by a factor of b equals to two. So then maybe you say that I will pick sigma i prime and u sigma i to be sigma i plus sigma i plus 1 over 2, doing some kind of an average. The problem with this choice is that if the two spins are both pluses, I will get plus. If they're both minuses, I will get minus. If there is one plus and one minus, I will get zero.

Why that is not nice is that that changes the structure of the theory. So I started with binary variables. I do this rescaling. If I choose this scheme, I will have three variables per site.

But I can insist upon keeping two variables per site, as long as I do everything consistently and precisely. So maybe I can say that when this occurs, where the two sites are different, and the average would be zero, I choose as tiebreaker the left one. So then I will have plus or minus.

Now you can convince yourself that if I do this, and I choose always the left one as tiebreaker, the story is the same as just keeping the left one. So essentially, this kind of averaging with a tiebreaker is equivalent to getting rid of every other spin. And so essentially what I can do is to say that I call a sigma i prime. Now, in the new listing that I have, this thing is no longer, let's say, the tenth one. It becomes the fifth one because I removed half of things. So sigma i prime is really sigma 2i minus 1.

So basically, all the odd ones I will call to be my new spins. All the even ones I want to get rid of, I'll call them SI. So this is just a renaming of the variables. I did some handwaving to justify. Effectivity, all I did was I broke this sum into two sets of sums, but I call sigma i prime and SI. And each one of them the index i, rather than running from 1 to n in this new set, the index i runs from 1 to n over 2.

So what I have said, agian, is very trivial. I've said that the original sum, I bring over as a sum over the odd spin, whose names I have changed, and a sum over even spins, whose names I have called SI. And I have an interaction, which I can write as sum over i, essentially running from 1 to n over 2. I start with the interaction that involves sigma i prime with si because now each sigma i prime is acting with an s on one side. And then there's another interaction, which is SI, and the next.

So essentially, I rename things, and I regrouped bonds. And the sum that was n terms now n over 2 pairs of terms. Nothing significant. But the point is that over here, I can rewrite this as a sum over sigma i prime. And this is a product over terms where within each term, I can sum over the spin that is sitting between two spins that I'm keeping. So I'm getting rid of this spin that sits between spin sigma i prime and sigma i plus 1 prime.

Now, once I perform this sum over SI here, then what I will get is some function that depends on sigma i prime and sigma i plus 1 prime. And I can choose to write that function as e to the b prime sigma i prime sigma i plus 1 prime. And hence, the partition function after removing every other spin is the same as the partition function that I have for the remaining spins weighted with this b prime.

So you can see that I took the original partition function and recast it in precisely the same form after removing half of the degrees of freedom. Now, the original b for the Ising model is going to be parameterized by g, h, and k. So the b prime I did parameterize. So this, let's say, emphasizes parameterized by g, h, k. I can similarly parameterize this by g prime, h prime, k prime.

And how do I know that? Because when I was writing this, I emphasized that this is

the most general form that I can write down. There is nothing else other than this form that I can write down for this. So what I have essentially is that this e to the b prime, which is e to the g prime plus h prime sigma i prime plus sigma i plus 1 prime plus k prime sigma i prime sigma i plus 1 prime involves these three parameters, is obtained by summing over SI. Let me just call it s being minus plus 1 of e to the g plus h sigma 1 sigma i prime plus SI plus k sigma i prime SI plus g plus kh sigma SI plus sigma i plus 1 prime plus k SI sigma i plus 1.

So it's an implicit equation that relates g prime, h prime, k prime, to g, h, and k. And in particular, just to make the writing of this thing explicit more clearly, I will give names. I will call e to the k to the x, e to the h to by, e to the g to bz. And here, similarly, I will write x prime e to the k prime, y prime e to the h prime, and z prime is e to the g prime.

So now I just have to make a table. I have sigma i prime sigma i plus 1 prime. And here also I can have values of s. And the simplest possibility here is I have plus plus. Actually, let's put this number further out.

So if both the sigma primes are plus, what do I have on the left hand side? I have e to the g prime, which is z prime. e to the 2h prime, which is y prime squared. e to the k prime, which is x prime.

What do I have on the left hand side? On the left hand side, I have two possible things that I can put. I can put s to be either plus or s to be minus, and I have to sum over those two possibilities. You can see that in all cases, I have e to the 2g. That's a trivial thing. So I will always have a z squared. Irrespective of s, I have two factors of e to the h sigma prime.

OK, you know what happened? I should've used-- since I was using b, this factor of h over 2. So I really should have put here an h prime over 2, and I should have put here an h over 2 and h over 2 because the field that was residing on the sites, I am dividing half of it to the right and half of it to the left bond. And since I'm writing things in terms of the bonds, that's how I should go. So what I have here actually is now one factor of i prime. That's better.

Now, what I have on the right is similarly one factor of y from the h's that go with sigma 1 prime, sigma 2 prime. And then if s is plus, then you can see that I will get two factors of e to the k because both bonds will be satisfied. Both bonds will be plus plus. So I will get x squared, and the contribution to the h of the intermediate bond s is going to be 1e to the h. So I will get x squared y. Whereas if I put it the intermediate sign for s to be minus 1, then I have two pluses at the end. The one in the middle is minus. So I would have two dissatisfied factors of e to the k becoming e to the minus k. So it's x to the minus 2. And the field also will give a factor of e to the minus h or y inverse.

So there are four possibilities here. The next one is minus minus. z prime will stay the way it is. The field has switched sign. So this becomes y prime inverse. But since both of them are minus-- minus, minus-- the k factor is satisfied and happy. Gives me x prime because it's e to the plus k.

On the right hand side, I will always get the z squared. The first factor becomes the inverse. Now, s plus 1 is a plus site, plus spin, that is sandwiched between two minus spins. So there are two unhappy bonds. This gives me x to the minus 2. Why the spin is pointing in the direction of the field. So there's the y here. And here I will get x squared and y inverse because now I have three negative signs. So all the k's are happy.

There's the next one, which is plus and minus. Plus and minus, we can see that the contribution to the field vanishes. Because sigma i prime and plus sigma i plus 1 prime are zero. I will still get z, the contribution, because plus and minus, the bond between them gives me e to the minus k prime. I have here z squared.

There is no overall contribution of y for the same reason that there was nothing here. But from s, I will get a factor of y plus y inverse. And there's no contribution for x because I have a plus and a minus. And if the spin is either plus or minus in the middle, there will be one satisfied and one dissatisfied bond. Again, by symmetry, the other configuration is exactly the same.

So while I had four configuration, and hence four things to match, the last two are the same. And that's consistent with my having three variables, x prime, y prime, and z prime, to solve. So there are three equations and three variables.

Now, to solve these equations, we can do several things. Let's, for example, multiply all four equations together. What do we get on the left hand side? There are two x's, two inverse x's, y, and inverse y, but four factors of z. So I get z prime to the fourth factor. On the other side, I will get z to the eight, and then the product of those four factors.

I can divide one, equation one, by equation two. What do I get? The x's cancel. The x's cancel. So I will get y prime squared. On the other side, divide these two. I will get y squared. x squared y plus x squared y inverse, x minus 2y plus x2 minus-- yeah, x2 y inverse.

And finally, to get the equation for x prime, what I can do is I can multiply 1 and 2, divide by 3 and 4. And if I do that, on the left hand side I will get x prime to the fourth. And on the right hand side, I will get x squared y plus x to the minus 2 y inverse x minus 2y x squared y inverse divided by y plus y inverse squared.

So I can take the log of these equations, if you like, to get the recurrence relations for the parameters. For example, taking the log of that equation, you can see that I will get g prime. From here, I would get 2g plus some function. There's some delta g that depends on k and h.

If I do the same thing here, you can see that I will get an h prime that is h plus some function from the log of this that depends on k and h. And finally, I will get k prime some function of k and h. This parameter g is not that important. It's basically an overall constant. The way that we started it, we certainly don't expect it to modify phase diagrams, et cetera. And you can see that it never affects the equations that govern h and k, the two parameters that give the relative weight of different configurations. Whether you are in a ferromagnet or a disordered state is governed by these two parameters. And indeed, we can ignore this.

Essentially, what it amounts to is as follows, that every time I remove some of the spins, I gradually build a contribution to my free energy. Because clearly, once I have integrated out over all of the energies, what I will have will be the partition function. Its log would be the free energy. And actually, we encountered exactly the same thing when we were doing momentum space RG. There was always, as a result of the integration, some contribution that I call delta f that I never looked at because, OK, it's part of the free energy but does not govern the relative weights of the different configurations. So these are really the two things that we need to focus on.

Now, if we did things correctly and these are correct equations, had I started in the subspace where h equals to zero, which had up-down symmetries, I could have changed all of my sigmas to minus sigma, and for h equals to zero, the energy would not have changed. So a check that we did things correctly is that if h equals to zero, then h prime has to be zero.

So let's see. If h equals to zero or y is equal to one, well if y is equal to one, you can see that those two factors in the numerator and denominator are exactly the same. And so y prime stays to one. So this check has been performed.

So if I am on h equals to zero on the subspace that has symmetry, then I have only one parameter, this k or x. And so on this space, the recursion relation that I have is that x prime to the fourth is x squared plus x to the minus 2 squared. I think I made a mistake here. No, it's correct.

y plus y inverse. OK, but y is one. So this is divided by 2 squared, which is 4. But let me double check so that I don't go-- yeah. So this is correct. So I can write it in terms of x being e to the k. So what I have here is e to the 4k prime. I have here e to the 2k plus e to the minus 2k divided by 2, which is hyperbolic cosine of 2k, squared. So what I will get is k prime is 1/2 log hyperbolic cosine of 2k.

Now, k is a parameter that we are changing. As we make k stronger, presumably things are more coupled to each other and should go more towards order. As k goes towards zero, basically the spins are independent. They can do whatever they

like. So it kind of makes sense that there should be a fixed point at zero corresponding to zero correlation length.

And let's check that. So if k is going to zero-- it's very small-- then k prime is 1/2 log hyperbolic cosine of a small number, is roughly 1 plus that small number squared over 2. There's a series like that.

Log of 1 plus a small number is a small number. So this becomes 4k squared over 2 over 2. It's k squared. So it says that if I, let's say, start with a k that is 1/10, my k prime would be 1/100 and then 1/10,000. So basically, this is a fixed point that attracts everything to it. Essentially, what it says is you may have some weak amount of coupling. As you go and look at the spins that are further and further apart, the effective coupling between them is going to zero. And the spins that are further apart don't care about each other. They do whatever they like. So that all makes physical sense.

Well, we are really interested in this other end. What happens at k going to infinity? We can kind of look at this equation. Or actually, look at this equation, also doesn't matter. But here, maybe it's clearer. So I have e to the 4k prime. And this is going to be dominated by this. This is e to the 2k. e to the minus 2k is very small. So it's going to be approximately e to the 4k divided by 4.

So what I have out here is that k is very large. k prime is roughly k, but then smaller by a factor that is 1/2 of log 2. So we try to take two spins that are next to each other, couple them with a very strong k. Let's say a million, 10 million, whatever. And then I look at spins that are too further apart. They're still very strongly coupled, but slightly less. It's a million minus log 2. So it very, very gradually starts to move away from here.

But then as it kind of goes further, it kind of speeds up. What it says is that it is true. At infinity, you have a fixed point. But that fixed point is unstable. And even if you have very strong but finite coupling, because things are finite, as you go further and further out, the spins become less and less correlated. This, again, is another indication of the statement that a one dimensional system will not order.

So one thing that you can do to sort of make this look slightly better, since we have the interval from zero to infinity, is to change variables. I can ask what does tang of k prime do? So tang k prime is e to the 2k prime minus 1 divided by e to the 2k prime plus 1. You write e to the 2k's in terms of this variable.

At the end of the day, a little bit of algebra, you can convince yourself that the recursion relation for tang k has a very simple form. The new tang k is simply the square of the old tang k. If I plot things as a function of tang k that runs between zero and one, that fixed point that was at infinity gets mapped into one, and the flow is always towards here. There is no other fixed point in between.

Previously, it wasn't clear from the way that I had written whether potentially there's other fixed points along the k-axis. If you write it as t prime tang is t squared, where t stands for tang, clearly the only fixed points are at zero and one. But now this also allows us to ask what happens if you also introduce the field direction, h, in this space.

Now, one thing to check is that if you start with k equals to zero, that is independent spins, you look at the equation here. k equals to 0 corresponds to x equals to one. This factor drops out. You see y prime is equal to y. So if you have no coupling, there is no reason why the magnetic field should change. So essentially, this entire line corresponds to fixed points.

Every point here is a fixed point. And we can show that, essentially, if you start along field zero, you go here. If you start along different values of the field, you basically go to someplace along this line. And all of these flows originate, if you go and plot them, from back here at this fixed point that we identified before at h equals to zero.

So in order to find out what is happening along that direction, all we need to do is to go and look at x going to infinity. With x going to infinity, you can see that y prime squared, the equation that we have for y, y prime squared is y squared. And then there's this fraction. Clearly, the terms that are proportional to x squared will

overwhelm those proportional to x to the minus 2. So I will get a y and a y inverse from the numerator and denominator.

And so this goes like y to the fourth. Which means that in the vicinity of this point, what you have is that h prime is 2h. So this, again, is irrelevant direction. And here, you are flowing in this direction. And the combination of these two really justifies the kind of flows that I had shown you before. So essentially, in the one dimensional model, you can start with any microscopic set of k and h's that you want. As you rescale, you essentially go to independent spins with an effective magnetic field.

So let's say we start with very close to this fixed point. So we had a very large value of k. I expect if I have a large value of k, neighboring spins are really close to each other. I have to go very far before, if I have a patch of pluses, I can go over to a patch of minuses. So there's a large correlation length in the vicinity of this point. And that correlation length is a function of our parameters k and h.

Now, the point is that the recursion relation that I have for k does not look like the type of recursions that I had before. This one is fine because I can think of this as my usual h prime is b to the yh h, where here I'm clearly dealing with a b equals to two, and so I can read off my yh to be one. But the recursion relation that I have for k is not of that form. But who said that I choose k as the variable that I put over here? I have been doing all of these manipulations going from k to x, et cetera.

One thing that behaves nicely, you can see, is a variable if I call e to the minus k prime is square root of 2 e to the minus k. At k equals to infinity, this is something that goes to zero on both ends. But if its k is large but not finite, it says that on the rescaling, it changes by a factor of root 2. So if I say, well, what is c as a function of the magnetic field, rather than writing k, I will write it as e to the minus k. It's just another way of writing things.

Well, I know that under one step of RG, all my length scales have shrunk by a factor of two. So this is twice the c that I should write down with 2h and root 2 e to the minus k. So I just related the correlation length before and after one step of RG, starting from very close to this point.

And for these two factors, 2 and root 2, I use the results that I have over there. And this I can keep doing. I can keep iterating this. After l steps of RG, this becomes 2 to the l, c of 2 to the lh, and 2 to the l over 2 e to the minus k.

I iterate this sufficiently so that I have moved away from this fixed point, where everything is very strongly correlated. And so that means I move, let's say, until this number is something that is of the order of one. Could be 2. Could be 1/2. I really don't care. But the number of iterations, the number of rescalings by a factor of two that I have to do in order to achieve this, is when 2 to the l is of the order of e to the k. e to the 2k, sorry.

And if I do that, I find that c should be e to the 2k some function of h e to the 2k. So let's say I am at h equals to zero. And I make the strength of the interaction between neighbors stronger and stronger. If I asked you, how does the correlation length, how does the size of the patches that are all plus and minus change as k becomes stronger and stronger? Well, RG tells you that it goes in this fashion. In the problem set that you have, you will solve the problem exactly by a different method and get exactly this form.

What else? Well, we said that one of the characteristics of a system is that the free energy has a singular part as a function of the variables that we have that is related to the correlation length to the power d. In this case, we have d equals to one. So the statement is that the singular part of the free energy, as you approach infinite coupling strength, behaves as e to the minus 2k some other function of h, e to the 2k.

Once you have a function such as this, you can take two derivatives with respect to the field to get the susceptibility. So the susceptibility would go like two derivatives of the free energy with respect to the field. You can see that each derivative brings forth a factor of e to the 2k. So two derivatives will bring a factor of e to the 2k, if I evaluate it at h equals to zero.

So the statement is that if I'm at zero field, and I put on a small infinitesimal

magnetic field, it tends to overturn all of the spins in the direction of the field. And the further you are close to k to infinity, there are larger responses that you would see. The susceptibility of the response diverges as k goes to infinity. So in some sense, this model does not have phase transition, but it demonstrates some signatures of the phase transition, such as diverging correlation length and diverging susceptibility if you go to very, very strong nearest neighbor coupling.

There is one other relationship that we had. That is, the susceptibility is an integral in d dimension over the correlation length, which we could say in one dimension-- well, actually, let's write it in d dimension. e to the minus x over c divided by x. We introduce an exponent, eta, to describe the decay of correlations. So this would generally behave like c to the 2 minus eta.

Now, in this case, we see that both the correlation length and susceptibility diverge with the same behavior, e to the 2k. They're proportional to each other. So that immediately tells me that for the one dimensional system that we are looking at, eta is equal to one. And if I substitute back here, so eta is one, the dimension is one, and the two cancels. Essentially, it says that the correlation length in one dimension have a pure exponential decay. They don't have this sub leading power load that you would have in higher dimensions. So when you do things exactly, you will also be able to verify that.

So all of the predictions of this position space RG method that we can carry out in this one dimensional example very easily, you can also calculate and obtain through the method that is called transfer matrix, and is the subject of the problem set that was posted yesterday. Also, you can see that this approach will work for any system in one dimension. All I really need to ensure is that the b that I write down is sufficiently general to include all possible interactions that I can write between two spins.

Because if I have some subset of those interactions, and then I do this procedure that I have over here and then take the log, there's no reason why that would not generate all things that are consistent with symmetry. So you really have to put all

possible terms, and then you will get all possible terms here, and there would be a recursion relation that would relate. You can do this very easily, for example, for the Potts model. For the continuous spin systems, it becomes more difficult.

Now let's say briefly as to why we can solve things exactly, let's say, for the one dimensionalizing model by this procedure. But this procedure we cannot do in higher dimensions. So let's, for example, think that we have a square lattice. Just generalize what we had to two dimensions.

And let's say that we want to eliminate this spin and-- well, let's see, what's the best way? Yeah, we want to eliminate a checkerboard of spins. So we want to eliminate half of the spins. Let's say the white squares on a checkerboard. And if I were to eliminate these spins, like I did over here, I should be able to generate interactions among spins that are left over.

So you say, fine. Let's pick these four spins. Sigma 1, sigma 2, sigma 3, sigma 4, that are connected to this spin, s, that is sitting in the middle, that I have to eliminate. So let's also stay in the space where h equals to zero, just for simplicity. So the result of eliminating that is I have to do a sum over s.

I have e to the-- let's say the original interaction is k, and s is coupled by these bonds to sigma 1, sigma 2, sigma 3, sigma 4. Now, summing over the two possible values of s is very simple. It just gives me e to the k times the sum plus e to the minus k, that sum. So it's the same thing as 2 hyperbolic cosine of k, sigma 1 plus sigma 2 plus sigma 3 plus sigma 4.

We say good. I had something like that, and I took a log so that I got the k prime. So maybe I'll do something like a recasting of this, and maybe a recasting of this will give me some constant, will give me some kind of a k prime, and then I will have sigma 1, sigma 2, sigma 2, sigma 3, sigma 3, sigma 4, sigma 4, sigma 1. But you immediately see that that cannot be the entire story. Because there is really no ordering among sigma 1, sigma 2, sigma 3, sigma 4. So clearly, because of the symmetries of this, you will generate also sigma 1 sigma 3 plus sigma 2 sigma 4.

That is, eliminating this spin will generate for you interactions among these, but also interactions that go like this. And in fact, if you do it carefully, you'll find that you will also generate an interaction that is product of all four spins. You will generate something that involves all of the force. So basically, because of the way of geometry, et cetera, that you have in higher dimensions, there is no trick that is analogous to what we could do in one dimension where you would eliminate some subset of spins and not generate longer and longer range interactions, interactions that you did not have.

You could say, OK, I will start including all of these interactions and then have a higher, larger parameter space. But then you do something else, you'll find that you need to include further and further neighboring interactions. So unless you do some kind of termination or approximation, which we will do next time, then there is no way to do this exactly in higher dimensions.

AUDIENCE: Question.

PROFESSOR: Yes.

AUDIENCE: I mean, are you putting any sort of weight on the fact that, for example, sigma 1 and sigma 3 are farther apart than sigma 1 and sigma 2, or are we using taxi cab distances on this lattice?

PROFESSOR: Well, we are thinking of an original model that we would like to solve, in which I have specified that things are coupled only to nearest neighbors. So the ones that correspond to sigma 1, sigma 3, are next nearest neighbors. They're certainly farther apart on the lattice. You could say, well, there's some justification, if these are ions and they have spins, to have some weaker interaction that goes across here. There has to be some notion of space. I don't want to couple everybody to everybody else equivalently. But if I include this, then I have further more complicated Hamiltonian. And when I do RG, I will generate an even more complicated Hamiltonian. Yes.

AUDIENCE: [INAUDIBLE].

PROFESSOR: Question is, suppose I have a square lattice. Let's go here. And the suggestion is, why don't I eliminate all of the spins over here, maybe all of the spins over here? So the problem is that I will generate interactions that not only go like this, but interactions that go like this. So the idea of what happens is that imagine that there's these spins that you're eliminating. As long as there's paths that connect the spins that you're eliminating to any other spin, you will generate that kind of couple.

Again, the reason that the one dimensional model works is also related to its exact solvability by this transfer matrix method. So I will briefly mention that in the last five minutes. So for one dimensional models, the partition function is a sum over whatever degree of freedom you have. Could be Ising, Potts, xy, doesn't matter. But the point is that the interaction is a sum of bonds that connect one site to the next site. I can write this as a product of e to the b of SI and SI plus 1.

Now, I can regard this entity-- let's say I have the Potts model q values. This is q possible values. This is q possible values. So there are q squared possible values of the interaction. And there is q squared possible values of this Boltzmann weight. I can regard that as a matrix, but I can write in this fashion. And so what you have over there, you can see is effectively you have s1, ts2, s2, ts3, and so forth. And if I use periodic boundary condition like the one that I indicated there so that the last one is connected to the first one, and then I do a sum over all of these s's, this is just the product of two matrices. So this is going to become, when I sum over s2, s1 t squared s3 and so forth. You can see that the end result is trace of the matrix t raised to the power of n.

Now, the trace you can calculate in any representation of the matrix. If you manage to find the representation where t is diagonal, then the trace would be sum over alpha lambda alpha to the n, where these lambdas are the eigenvalues of this matrix. And if n is very large, the thermodynamic limit that we are interested, it will be dominated by the largest eigenvalue.

Now, if I write this for something like Potts model or any of the spin models that I had indicated over there, you can see that all elements of this matrix being these

Boltzmann weights are plus, positive. Now, there's a theorem called Frobenius's theorem, which states that if you have a matrix, all of its eigenvalues are positive, then the largest eigenvalues is non-degenerate. So what that means is that if this matrix is characterized by a set of parameters, like our k's, et cetera, and I change that parameter, k, well the eigenvalue is obtained by looking at a single matrix. It doesn't know anything about that.

The only way that the eigenvalue can become singular is if two eigenvalues cross each other. And since Frobenius's theorem does not allow that, you conclude that this largest eigenvalue has to be a perfectly nice analytical function of all of the parameters that go into constructing this Hamiltonian. And that's a mathematical way of saying that there is no phase transition for one dimensional model because you cannot have a crossing of eigenvalues, and there is no singularity that can take place.

Now, an interesting then question or caveat to that comes from the very question that was asked over here. What if I have, let's say, a two dimensional model, and I regard it essentially as a complicated one dimensional model in which I have a complicated multi-variable thing over one side, and then I can go through this exact same procedure over here also? And then I would have to diagonalize this huge matrix. So if this is l, it would be a 2 to the l by 2 to the l matrix.

And you may naively think that, again, according to Frobenius's theorem, there should be no phase transition. Now, this is exactly what Lars Onsager did in order to solve the two dimensionalizing model. He constructed this matrix and was clever enough to diagonalize it and show that in the limit of l going to infinity, then the Frobenius's theorem can and will be violated. And so that's something that we will also discuss in some of our future lectures. Yes.

AUDIENCE: But it won't be violated in the one dimensional case, even if n goes to infinity?

PROFESSOR: Yeah, because n only appears over here. Lambda max is a perfectly analytic function.