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PROFESSOR: We want to start talking about a seemingly simple but very complex system in physics, the harmonic oscillator. So the next part is actually due to Professor Vladan Vuletic, who worked out the topic very nicely about, how precisely can you measure frequencies?

And I don't need to remind you that some of the most accurate measurements in all of physics are done by measuring frequency. It's actually a kind of unwritten rule. If you want to measure something precisely, make sure that you find a way that this quantity can be measured in a frequency measurement. Because frequencies, that's what we can measure-- with synthesizers, with clocks, and such. So therefore, the question, how precisely can you measure frequency, is a question which is actually relevant for all precision measurements.

Well, we know we have Fourier theorem. If you have an oscillator which oscillates for a time Δt , we have the Fourier theorem which says we have a finite width of the frequency spectrum $\Delta \omega$, or there is the spread of frequency components involved in such a way that $\Delta \omega \Delta t$ is large or equal than $1/2$. The case of $1/2$ is realized for Gaussian wave packets.

And of course, Fourier theorem should also-- this Fourier limit should remind you of Heisenberg's uncertainty relation. Of course, Fourier limit and Heisenberg uncertainty relation are related because what Heisenberg expressed turns out to be simply the limit due to the wave nature of matter.

OK. So I brought the clicker because I want you to think about some seemingly simple question. The first question is whether the uncertainty $\Delta \omega \Delta t$ larger than $1/2$, does this uncertainty hold for purely classical systems?

So think about it and answer in your clicker. A is 1 and B-- oh, I should say that.

So we now assume we have a purely classical-- you can even think a mechanical, large mechanical object, purely classical harmonic oscillator. You can observe it for time Δt , or it oscillates for time Δt . Can you determine its frequency better than this uncertainty suggests?

19, 20. Just make up your mind. As I said, your responses are not recorded, so nothing to risk. OK.

Yes. OK, the majority gave the right answer. The situation is that the answer is yes, if you have a good signal to noise ratio.

So what happens is, if you-- we have to bring in the fact that we have noise. So if you have a wave formed, there may be noise around it. And if you look at the spectrum, the spectral components have a certain width. And this width, $\Delta\omega$, is given by Fourier's theorem by the time Δt you had for observation.

But as you see, you can determine the center of this spectral peak with an accuracy $\Delta\omega$, which may be much better than the [width] $\Delta\omega$. And the rule of thumb is that you can split a line by your signal to noise ratio. So typically, the accuracy of a measurement is whatever the width of the spectrum is, and then you can split the line by the signal to noise ratio. This is called splitting the line.

Factor of hundred is usually regarded as straightforward. But if you want to go to larger than that, it becomes a challenge. Because even if you have a very good signal to noise ratio, you really have to make sure that you know the line shape, that you know that it is, for instance, symmetric, and that the line center of the observed shape is really where the frequency is.

I will give you some examples in a few moments. But before that, I would like to continue our discussion whether we can measure the angle of frequency to better than the Heisenberg limit in case of a quantum-mechanical oscillator. So the question is, can you measure the frequency of a quantum-mechanical harmonic oscillator in a time Δt to an accuracy which is better than the limit of

Heisenberg's uncertainty? OK, your vote, please.

All right. OK. Why don't we hold that for a second and proceed to the next question which is the same? But now, instead of a quantum-mechanical harmonic oscillator, we take something we are very familiar with-- an optical laser-- and we observe a laser pulse lasting a duration Δt . So same situation, but instead of observing a quantum-mechanical harmonic oscillator, we observe a laser pulse. For the laser, can we measure the frequency of the optical radiation better than this equality tells us?

OK. At least people are consistent because the first thing I wanted to tell you is that, is the laser actually a classical harmonic oscillator or quantum harmonic oscillator?

Well, we use a quantum description of light and the laser is the population of photons in the single mode of the electromagnetic field. So in that sense, the laser is fully quantum. But in the limit that the laser is many, many photons-- and some of you know about coherence states. If there's a coherent state with a large photon number, the laser is actually the classical limit of an electromagnetic field.

So maybe that tells me that the answers to questions 2 and question 3 should probably be the same. And I want to say more about it. So at least in this class, you were consistent. I have often seen a big discrepancy in the answers between question 2 and question 3.

OK, so the answer is yes. So both for the laser-- I first explain the laser to you, and then we go back to the pure quantum system, things have actually certain subtleties.

But for the laser, it is obvious, at least if I tell you how I want to measure it, because I can take the laser and create a beat note with another very stable laser. And I record this beat note on a photodiode.

I can realize by making this other laser, the local oscillator stronger and stronger, I can create a beat note which is larger and larger and corresponds to a macroscopic electric current which can be measured with very high precision.

So you can realize an arbitrarily high signal to noise ratio by using a strong local oscillator. And then you can actually say the photocurrent, which comes out of the photodiode, is actually-- you can regard this microscopic current as purely classical. And then of course, the answer to the first question applies. So that takes care of question 3 by mapping it actually on question 2.

But now by saying that the laser also has a quantum-mechanical limit and I'm not changing anything, we realize that probably the answer to question 2 should also be yes. So let's, therefore, ask our self, what is the situation when the Heisenberg uncertainty relation applies?

Well, one is we have to be really careful. It predicts the outcome of a single measurement on a single quantum system. Let me write that down.

Or if the Heisenberg uncertainty relation sets a limit, how well we prepare a quantum system. It's about a single quantum system, and then we perform a single measurement.

So in a sense, if we would say all you have is a single photon, which is a very special quantum system. You have a single photon and you measure the frequency of the photon only once. Then you will find the limit, which is the Heisenberg limit. You cannot, with a single measurement on a single photon, determine the accuracy of the frequency better than this.

And of course, you can get higher accuracy by doing repeated measurements or by using many photons. We talk about it more in 8.422, but I just want to remind you, if you have n uncorrelated photons. In other words, we perform n measurements on n different objects, then the signal to noise ratio is-- just by Poisson distribution, square root n . And therefore, the resolution for the frequency of the photons is better than the Heisenberg or the Fourier limit by 1 over the square root n .

Some of you-- and actually, in Professor Vuletic's group, there is research on it that if you have correlated-- well, in his case, correlated atoms. But if you had correlated photons, then you can even do better. You can reach what is sometimes called the

Heisenberg limit where you are better than the limitation given by Fourier's theorem or by the Heisenberg uncertainty relation by a factor of $1/n$.

OK. So as far as the question 2 where the quantum harmonic oscillator is concerned, we would say the answer is yes, if you have a single photon at frequency ω_0 , which interacts with the quantum harmonic oscillator at frequency ω_0 .

However, the answer would, even in that case, be no if you have harmonic oscillator levels and you take a photon and by a nonlinear process excite the n -th level.

So you have a single photon now. You can resolve the energy ΔE of this level. A single photon, a single quantum object, you can define the energy-- that's Heisenberg's uncertainty relation. The energy is determined to that precision. But your frequency ω of the optical pulse is n times ω_0 using nonlinear process. And then you can determine the frequency of the harmonic oscillator, even for a single quantum system and a single photon with a precision which is $1/n$ times better. So you have to be also careful, but I don't want to beat it to death now, to distinguish between the accuracy at which Heisenberg's uncertainty relation maybe limits the measurement of an energy level. And how this is related to the frequency of the harmonic oscillator.

And by going sort of immediately to the n -th level, you can, of course, measure the distance between two levels more accurately because you have increased your [INAUDIBLE]. Any questions?

AUDIENCE: Sorry, I just got a little confused about when yes means one thing and no means one thing. So you're saying that you can beat the uncertainty relation in questions 1, 2, and 3 if you can put it in a way where you get good signal to noise?

PROFESSOR: OK, I gave you-- sorry for being complicated, but the physics is complicated. I try to give it to you in different layers. I first looked at the classical limit, which is pretty clear-cut. Then, I used a laser. The laser has a classical limit where the answer is the same as in the classical limit. But then we can talk also in the laser in the limit of

single photons.

And then I said, OK, the single photon interacting with a single quantum system, this is really when it is quantum. And if you have a two-level system and you [INAUDIBLE] it with a single photon, then you can make a measurement which is limited by this inequality.

But then I said there is a caveat. And this is if you bring in a nonlinear process.

So if you bring in a nonlinear process, we can go up n steps. We can drive-- we can, by some nonlinearity, drive the harmonic oscillator from the ground state to the n state. Then, everything we have said about a single photon and the measurement of the resonance and such applies to this photon. But the energy level of the harmonic oscillator has now been measured with n times higher precision because we can divide by n .

So the answer is yes, yes, then quantum mechanics we cannot make it more accurate unless we pull some tricks. And nonlinear physics would be a trick.

So in general, the situation where you really limit it by this inequality where your precision is limited would really only apply to a single photon, a single quantum physics, and linear physics. Other questions?

AUDIENCE: Yes. Maybe just the phrasing.

AUDIENCE: What does Δt mean for a single photon measurement?

PROFESSOR: Δt could be the time you allow yourself to make the measurement. You have a measurement apparatus. You switch it on, you switch it off. Eventually, you want to get out of grad school. I mean, you don't want to take an infinite amount of time for the measurements. There's always a window, Δt , and there's a fundamental limit. The duration of the measurement limits the precision of the measurement.

OK. The next thing I want to discuss is the analogy, but also the differences between a harmonic oscillator and a two-level system. So what is a two-level system?

Well, it's a system with two levels. What is a harmonic oscillator?

Well, it's a system which has an infinite number of equidistant levels. I will tell you tell you later in this course when we talk about the AC and DC Stark effect, you talk about the polarizability and light scattering that you can regard the atom or the electron in the atom as a harmonic oscillator. An atom scatters light exactly in the same way as the charge which is connected to some support structure with a spring. How an oscillating charge would scatter light?

Well, you know of course, the atom is a two-level system. And the sort of model I make for the electron as a harmonic oscillator at a single resonance frequency, which is 100% exact in the limit of [? low ?] laser power.

Well, this realize is a harmonic oscillator. So therefore, what I'm telling you by this example, that there are situations where a two-level system and a harmonic oscillator are the same. Or, create the same [? observance, ?] create the same physics. Do you have any idea when the two systems may look the same or when the two systems react exactly in the same way to, for instance, external radiation?

AUDIENCE: At very low temperatures?

PROFESSOR: At very low temperature? Well--

AUDIENCE: [INAUDIBLE].

PROFESSOR: Yes. Well, we assume these are atoms and we always start in the common state. So let's assume we have 0 temperature. We have an atom. And maybe what I'm asking is, if I excite the atom and I said, there may be a situation where the atom is a two-level system but it reacts like a harmonic oscillator, when does it break down?

AUDIENCE: When the [INAUDIBLE].

PROFESSOR: Exactly. When we go beyond the perturbative limit when we use a strong excitation. So in other words-- and I like to give you the answer before I give you the full explanation, which now comes. If you start out in the ground state, you can see at 0

temperature, we have mainly all the population there.

If you start now driving the system, we put-- and I will say a little bit more about it. A little bit into the excited state. But it is the nature of a harmonic oscillator when you put something into the excited state that immediately a little bit goes into the second excited state. And this is, of course, something which you can only do in a harmonic oscillator but you cannot do in a level system. So to the extent that we have weak excitation and we can neglect the excitation in higher levels. To that extent, a two-level system and harmonic oscillator are identical.

Actually, what I'm saying appears trivial, but I really want you to think about it. It's actually a very profound statement. When can you describe a quantum-mechanical system as a harmonic oscillator?

For weak excitation when all what matters is that you have put a small fraction of the system into the first excited state. And you immediately realize that the feature which distinguishes a two-level system from a harmonic oscillator is the phenomenon of, let's say, saturation. You cannot go higher.

If you do not saturate a two-level system, it behaves like a harmonic oscillator. And therefore, it behaves completely classical.

OK, let's work that out a little bit. I'll give you some examples.

So the phenomenon of a two-level system is it has saturation. The maximum energy you can put in is one quantum. Whereas, a harmonic oscillator can never be saturated. Just think of the harmonic oscillator potential parabola. You can drive the system as high as you want. So you can go in this classical language to arbitrarily large amplitudes.

So what I just mentioned where the equivalence holds-- I always want you to have an example in mind-- is the Lorentz model for an atom where you describe the atom as an electron connected with a spring to the nucleus. And as we will see in a few weeks, this model gives the identical answer, identical to the quantum-mechanical

treatment for probabilities like the polarizability and the index of refraction for gas of atoms or molecules.

So if you have a two-level system, we can often think we have an s, ground state, and then p, excited state. And if you do a weak excitation, we have sort of a wave function, which is all, lets say, positive one side-- one sign. And then we [INAUDIBLE] a p orbital, which has a node which is positive and negative. And now we have the positive-negative and the positive and the-- there is the resonance frequency between the two. And that together results in an oscillating dipole.

So the simple model of superimposing an s state and a p state at a certain frequency gives us an oscillating dipole, which is the realization of a harmonic oscillator. But the harmonic oscillator, it oscillates.

And this is, of course, valid for sufficiently small excitation. So the question we have already addressed in the discussion, what is small?

So "small" means population of higher-excited states is negligible. So in other words, as long as the excitation of the first excited state is small, then we can neglect the excitation in the second excited state, which is even smaller. So let me kind of bring out the difference between a two-level system and a harmonic oscillator a little bit more by discussing the situation of cavity QED.

Let's assume we want 100% population in the first excited state. If you have a harmonic oscillator and the system is prepared in the first excited state, this is also called Fock state with one quantum of excitation. And it's a rather special state where people have worked hard to generate it because you cannot realize it in a harmonic oscillator. And let me sort of explain that in the following way.

If you have a harmonic oscillator, you start and you would drive it. And you try to put 100% in the $n = 1$ state. Before you have accumulated 100% in the $n = 1$ state, you drive it already to higher states. And of course, you know when you start driving a harmonic oscillator, classical or quantum-mechanically, you create a coherent state, which is a superposition of excited states.

So we would say an n equals 1 state cannot be excited. We usually get a coherent state, which is a superposition of many, or at least several, states. Whereas in a two-level system, we can just do a [INAUDIBLE] pulse. And to put all the atoms in the excited state is nothing special. Whereas, to have a cavity filled with photons and selectively excite the n equals 1 state, this is special because it's not easy. It's not straightforward.

So in cavity QED, you can do it if you have anharmonicities or nonlinearities. So let me explain that.

Well, it's an anharmonicity or some form of-- so if you have a situation where you have your harmonic oscillator, but the energy levels are not equidistance. So the difference between this first and second excited state are not the same, then you can drive the system. You can prepare Fock state in n equals 1 like in a two-level system. And you're out of resonance to drive it to higher states.

So here, what you utilize is a sort of two-level system. And that allows you to create those special state which are regarded as non-classical, very special states of the harmonic oscillator.

And one way how you can create it-- well, if you have an empty cavity, each photon has the same energy. Then you have an equidistant harmonic oscillator. But if you put, for instance-- you add an atom to the cavity, and the radiation is interacting with the atom, then you'll get-- we'll talk about it later. The atom and the photons interact with the Rabi frequency, and then you get a splitting, called the normal mode splitting. And this level splitting is proportional to the Rabi frequency. And we'll discuss it later, but many of you know that the Rabi frequency scales with the square root of the photon number.

So therefore, you have a splitting which is proportional to square root 1, square root 2, square root 3. And you have a spectrum which is no longer an equidistant system. And then you can create non-classical states of the photon field, non-classical states of a harmonic oscillator.

So anyway, I thought I wanted to bring it up at the beginning of the class, because a lot what we are discussing in this class is we'll rediscover in many situations-- in atoms, in the light, in the way how light and atoms interact, harmonic oscillators and two-level systems.

Often, I say they are the same. They behave in the same way. But I hope this introductory [? mark ?] tells you, when can you think in one limit and when do you have to apply the other limit? Any questions about that? Yes, Nancy.

AUDIENCE: So are we saying here that without changing the [INAUDIBLE] of the harmonic oscillator, we cannot use it as a harmonic oscillator? Like [INAUDIBLE]? Because when you put an atom in that [INAUDIBLE] changed, it was no more a harmonic oscillator. The levels changed.

PROFESSOR: Maybe all I'm saying is this is a pure harmonic oscillator. And in a pure harmonic oscillator, I think it's-- I don't know a proof of it, but it seems impossible to prepare a system in the first excited state because every attempt to put an excitation into the system would carry it higher up.

You would create a wave packet. You would create a superposition. So you have to do some thing. You have to break the degeneracy of the spectrum of the harmonic oscillator.

Of course, what you can do is you can put in an atom. You can use the atom as an aid to just put in exactly one photon into your cavity, and then you can remove the atom. Then you are back to an ideal harmonic oscillator, but you have overcome the limitation of the harmonic oscillator in preparing certain states.

Another take-home message you may take from this discussion is harmonic oscillators-- yes, we have quantum harmonic oscillators. But even the quantum harmonic oscillator follows a classical description. So the real quantumness-- what makes quantum optics quantum optics and cavity QED a wonderful example of quantum physics is the physics embedded in a two-level system. That we can put one quantum excitation into something, exactly one is as much quantum as you can

get.

This is realized in a two-level system and this is related to the phenomenon of saturation. You can saturate a two-level system, but you cannot saturate a harmonic oscillator. So with that, let me make the transition to another simple system. And we want to spend some time on it. And these are rotating systems.

So a system which rotates. Well, what do you think? Will it behave, using the discussion we just had, more like a classical system or more like a quantum-mechanical system?

Of course, I gave you a very special definition. What brings out quantum mechanics in a system?

The harmonic oscillator is always linear. You can drive it as hard as you want. You drive it hundred times stronger and the reaction is hundred times more. Everything is linear. The quantumness of a two-level system comes from saturation. What about a rotating system? Something which can go in a circle.

AUDIENCE: [INAUDIBLE]. So you're not going to have this degeneracy [INAUDIBLE].

PROFESSOR: OK, very good. You're immediately applying what is the spectrum. The spectrum is not equidistant, so it should bring in a difference. It's sometimes hard to ask a simple question without giving the answer away. But what I had in mind was a gyroscope, a gyroscope which is precessing.

And what I wanted to sort of lead you with the question is, if you have something which rotates, the amplitude is limited. A rotating object, let's assume a magnetic, classical magnetic moment. It can have a precession angle which is 180 degrees, but that's a maximum.

In other words, the rotating system when you excite it has a maximum amplitude, exactly as a two-level system. So that's what a rotating system and a two-level system-- actually then, I'm now specializing on more rotating gyroscope. If you have a free rotator, this, of course, can rotate with [INAUDIBLE] angular momentum and

the excitation spectrum would not be bound. So I think I have to rephrase the question the next time I teach the class.

I wanted to ask you here about a special rotating system, which is a precessing gyroscope. So rotating system.

If you think about precessing gyroscope, it has a bound on the amplitude it can be excited. So what I want to show you, today and in the next lecture, is the motion of classical magnetic moments.

When you think about the motion of classical magnetic moments, think about a compass needle, a magnetized needle which has angular momentum. And then, the system is acted upon with a magnetic field. So this is our system. And if angular momentum, magnetic moments, and torque come into play, we have the physics of classical rotation.

But the excitation spectrum here is limited because you can flip a compass needle and this is a maximum excitation. When the North Pole points in the opposite direction, that's the maximum excitation you can give it. So therefore, it has a limited amplitude of its excitation, unlike a harmonic oscillator. And at this point you may say, but maybe somewhat similar or analogous to a two-level system.

But the surprising result is-- at least it was surprising when I first learned about it. That it's not just somewhat analogous to a two-level system, it actually captures exactly a lot of the properties of the dynamics of a two-level system. So let me write that down.

The motion of classical magnetic moments provides a model. It's actually an exact model which captures essentially all features of the quantum mechanical two-level system. I want to show you today and the next lecture the concepts or Rabi frequency, of generalized or resonant Rabi frequency, all of that you find in the classical motion of a magnetic moment.

Or, for instance, the physics of rapid [INAUDIBLE] following, [INAUDIBLE]. A lot of physics we would usually associate with the quantum system, we find it here in a

purely classical system.

What aspects of the two-level system will we not find? Any ideas? Will?

AUDIENCE: Spontaneous emission?

PROFESSOR: Spontaneous emission, definitely. Yes. But actually, in a two-level system, in a quantum-mechanical two-level system, which we drive with a single frequency, spontaneous emission is also missing. Spontaneous emission, as we will discuss later, only comes into play when we say the excited state of the system interacts with many, many modes and not just the one mode we apply.

And typically, if you go to high frequency, we have an optical oscillator, we cannot avoid spontaneous emission. Whereas, for a quantum-mechanical spin $1/2$ interacting with microwaves, we can completely eliminate spontaneous emission. So spontaneous emission, I would say, comes into play at high frequency. So that's correct. But there is one aspect even at low frequency, one aspect of quantum mechanics which we cannot capture.

AUDIENCE: Having different G in a magnetic field?

PROFESSOR: Different G factors, yes. That's, as we see, a more quantitative aspect. But there is one very important feature about quantum mechanics you will never get in a classical system.

AUDIENCE: Spin?

PROFESSOR: Spin.

AUDIENCE: [INAUDIBLE].

AUDIENCE: [INAUDIBLE].

PROFESSOR: OK. I think you're skirting around. It's a quantum measurement process and projection. If you perform a measurement on a compass needle, it can be at any angle. But if you do a measurement on a quantum system, you do a projection.

After the measurement projects a system in either spin up or spin down. So the probabilistic nature, the projection occurring in a quantum measurement is, of course, absent in a classical system.

But when you say spin $1/2$ and quantum levels, this is sort of implied in it. If there are only two levels, there's only up and down and not an infinite number of angles. So what we will actually see is that we find an exact analogy between the classic system and the quantum mechanical system when we compare expectation values. But the individual measurement, the individual quantum measurement because it is projective is different.

OK, with that motivation, we are now talking about magnetic resonance. And we will later do a fully quantum mechanical description, but to get the concepts and also understand the analogies to a classical system, we want to understand what happens when we have a classic magnetic moment in magnetic fields. And that includes static fields, but then we want to excite the system. We want to drive the system, and this is time-varying fields. And we will assume that the fields are spatially uniform.

So let me just remind you of the obvious equations of motion that also allows me to introduce the nomenclature. The interaction energy between a classical magnetic moment μ and the magnetic field is $\mu \cdot B$. The force is the gradient of the interaction energy, but it is 0 for uniform fields. So therefore, we don't need to look at the force. But the next thing which we then have to consider is the torque.

And when we think about the classical magnetic moment, you can think about a compass needle. But magnetic materials are complicated. If I think about the simplest magnetic moment, I think about a loop of current I and area A . And that's sort of the classical model for magnetic moment. So we have a magnetic moment μ .

And if we now add a magnetic field, which is at an angle, we have a torque. But just to make sure the torque is something which is nothing else than the Lewin's force on the electrons. But since electron is forced to go in a circle, we don't have to look

at the Lewin's force microscopically. We just immediately jump to the torque. And the torque is what describes the dynamics of the system. So we have torque.

When we have torque, we want to formulate the problem in terms of angular momentum. And our equation of motion is the classical equation of motion that the derivative of angular momentum is given by that.

Now, what makes those equations immediately solvable-- and to find the very easy limit is that the magnetic moment of the system we assume is proportional to its angular momentum.

Well, if you have a mechanical object which goes in a-- if you have a charged object which circles around a central potential, then you, of course, find immediately that if it moves faster, it has more angular momentum. It has a larger magnetic moment. So we use that as the defining equation for what is called the gyromagnetic ratio. Which, of course, is very closely related to G factors, which we define later on for atoms.

The gyromagnetic ratio is the ratio between magnetic moment and angular momentum. And then, we find that the derivative of angular momentum is given by this equation. And this is now an equation which you have seen in classical mechanics and in many situations. The solution of that is a pure precession. The motion is pure precession of the angular momentum around the axis of the magnetic field.

So in other words, we have the axis of the magnetic field. We have the angular momentum. And at a constant tipping angle, we have the tip of the angular momentum precesses around the magnetic field. And the precession happens with an angular frequency which is called the Larmor frequency.

The Larmor frequency, the frequency of precession, is proportional to the magnetic field and the gyromagnetic ratio. So let me give you an example for an electron. The gyromagnetic ratio is 2π times 2.8 megahertz per Gauss.

And we've discussed last class what it means when I take out 2π . Because the

Larmor frequency is an angular frequency. And angular frequency is not measured in Hertz because there is a 2π factor. And I just make it obvious where the 2π factor is hidden.

Now, this is for the electron. But if you have an ensemble of classical charges, an [INAUDIBLE] distribution of classical charges-- well, with the same charge to mass ratio, you find that the gyromagnetic ratio is $1/2$ of that. And this here is the Bohr magneton, which we will use quite often in this course.

The third example is the proton. The proton is heavier, has a heavier mass. About 1,000 times heavier than the electron. And therefore, the Larmor frequency is not megahertz per Gauss, it is kilohertz per Gauss. Any questions? These are more definitions and setting the stage.

Let me make a note. It's one of the many notes I will make in this course about factors of 2. There are factors of 2. If you miss it, you qualitatively miss the physics.

And let me in that context by talking about precession frequencies and magnetic moment, explain a factor of 2 which is related to the G factor of the electron.

So if you have the electron which has spin of $1/2$, in units of the Bohr magneton, what is the magnetic moment of the electron? $1/2$, 1, or 2? What is the magnetic moment of the electron? $1/2$, 1, or 2?

AUDIENCE: 2.

AUDIENCE: [INAUDIBLE].

PROFESSOR: I should have a clicker question on there. No. It's 1. 1 Bohr magneton. And let me sort of show the level structure of it.

This is [INAUDIBLE] energy. You have spin up and you have spin down. The difference is 2.8 megahertz per Gauss. And if you ask, what is the precession frequency of an electron in a magnetic field? It's 2.8 megahertz if the field is 1 Gauss.

And if you want to drive the rotation, if you want to change the precession angle-- we'll talk about that in great detail-- you better drive the system at 2.8 megahertz. But 2.8 megahertz is the difference of plus 1.4 and minus 1.4.

And therefore, the energy of the electron in a magnetic field is either plus or minus 1.4 megahertz per Gauss, and 1.4 is 1 Bohr magneton. The magnetic moment of the electron is 1 Bohr magneton. So precesses at 2.8.

OK, but let us contrast this with a classical current distribution, which has 1 unit of \hbar , which means the magnetic moment is 1 Bohr magneton exactly as the electron has.

Well, quantum mechanically means it has three different level-- minus 1, [INAUDIBLE] 0, and 1 because it has 1 unit of angular momentum. Since the system has 1 Bohr magneton, when the system stands up or stands down, the difference between spin up and spin down is 2.8 megahertz per Gauss.

OK, my question now is, what is the precession frequency of this classical charge distribution which has 1 unit \hbar of angular momentum? And I've just shown you the level structure.

If you will now create a wave packet of those three levels, which means-- a wave packet of the three levels means you have a spin which points in one direction. What is the precession frequency of that system?

Let's have a clicker question. So what is the precession frequency? Oops, what happened? So let me give you three choices. 2.4, 1.4, or 0.0 megahertz per Gauss. So please vote for A, B, or C.

Yes, it's 1.4. So they have the same magnetic moment. Quantum mechanically, you would see here is a G factor of 2. Here's a G factor of 1. But the easiest explanation is it precesses. The precession is a beat note between two energy levels.

Here, you have the beat note between those two energy levels happens at 1.4 megahertz. Therefore, when you want to drive it with an external radiation, you

have to drive it at 1.4 megahertz. You want to drive it level by level.

Whereas, this system has a beat note between two levels, and the difference is 2.8 megahertz. Anyway, whenever you get confused about factors of 2 with magnetic moment and precessing system, just think about those two simple examples. They have all the factors of 2 hidden in the simplest example possible.

All right. We have a rotating system. have a system which precesses. So we want to learn about rotations in general. And what I want to show you is that under very general circumstances, we can solve the equation, the dynamics of the system, by going into rotating frame.

You all know about rotating wave [INAUDIBLE] rotating wave approximations that's in quantum physics. I'm simply talking about a classical system and I want to solve the equations for the classical system by going to rotating frame. And I want to show you where this is exact and where not.

OK, so this is actually something which we do in undergraduate, in 8.01-- definitely, in 8.012. But let me remind you when we have a rotating vector which rotates with a constant angular frequency, then the time derivative of the vector is the cross product. But now we want to allow-- so this is when the vector is constant and it just rotates. But now, we want to assume that there is something else. There is an arbitrary time dependence of the vector in the rotating frame. So we have a vector which changes according to \dot{A} [INAUDIBLE]. But it also rotates.

And that means-- and this is exactly shown in classical mechanics that in the inertia frame, the time derivative is the sum of the two. It is the change of the vector in the rotating frame plus $\omega \times A$. So it has this equation. Has the simple two limiting cases that if there is no change in the rotating frame, then we retrieve the kinematics of pure rotation.

When our rotating frame is not rotating or it rotates at 0 angular frequency, then of course the two time derivatives are the same. But anyway, what I derived for you is an operator equation that the time derivative in the rotating frame is related to the

time derivative in the inertia frame in this way. And now we want to apply it to our angular momentum L dot.

So this is just applying the operator equation to our angular momentum L . And now we want to specialize that-- we just discussed that the time [INAUDIBLE]. I'm just looking for a sign problem, but it's sometimes hard to fix sign problems at the board.

The creation of motion for the angular momentum was that it's L cross γB .

Oh, I changed the order. There is no sign problem. And then, I add this. So if we now describe our precessing, classical magnetic moment, which has the equation of motion that L dot is L cross γB , if you describe it in a rotating frame, then the equation of motion gets modified as follows.

Now, what happens is we-- let me factor out the γ . γL cross B is the real field. So what we observe is that when we go into a rotating frame-- and this is exact-- that the real magnetic field gets replaced by an effective magnetic field. Because there is an extra term added to it, which we can call a fictitious magnetic field.

So this is just an exact transformation of our equation of motion for a precessing system into the rotating frame. And now, of course, we haven't made any assumptions what the rotating frequency is. But if you would choose the rotating frequency to be the Larmor frequency minus γ times B , then our effective magnetic field vanishes.

And then we know, because there is no magnetic field, that the angular momentum is constant in the rotating frame. In other words, the dynamics of the system means that L is constant in a rotating frame. And if you want to know what happens in the original, in the [? lab ?] frame, in the inertial frame, we just have to rotate back.

OK, that's something we want to take advantage of. But we fully apply it in the next class. I have only a few minutes left today, and I want to spend those few minutes to talk about another factor of 2.

Now, let me ask you the following. If you have an electron in a magnetic field, well, you know that the electron goes in circles. It's the cyclotron motion of the electron.

Now, just give me one second. I forgot to mention something for the classical system. For a classical charge distribution, the Larmor frequency is the charge of the particle. In case of the electron, it's e . Divided by $2m$ times B . So the Larmor frequency is e over $2m$ times B .

So therefore, we know that when we have an [INAUDIBLE] of positive and negative charges, and there is an effective magnetic moment, that this magnetic moment would precess at the Larmor frequency which is given by this expression. So who knows what the frequency of the cyclotron motion is? So when we have a free electron, what is the frequency at which it revolves? At which it goes in circles?

AUDIENCE: 2 times the Larmor frequency.

PROFESSOR: It's two times the Larmor frequency.

OK, I just wanted to mention it. There's an important factor of 2 which you should know about.

In previous classes, I spent 10 or 20 minutes to teach you about a few of them which is called Larmor's theorem. But I summarized the argument on the atomic physics wiki and I can't say more here in class than I've written on the wiki. So please read on our atomic physics wiki about Larmor's theorem.

Larmor's theorem shows you that under certain assumptions, you can transform away the effect of a magnetic field by going to the Larmor frequency. That looks exactly like what we have discussed here. But what we discussed here was exact. There was no approximation. Whereas, the derivation of Larmor's theorem, which talks about charge distributions-- not about magnetic moments, about charge distributions-- has to make certain approximations.

So just want to point your attention to that there are two derivations about Larmor frequency. One is exact, which I gave to you. There is another one which is

Larmor's theorem, which applies to isolated charges which is not exact. But they both conclude that you can transform away the effect of a magnetic field by going to rotating frame at the Larmor frequency.

And the fact that Larmor's theorem is not exact is actually illustrated by this example of a free electron where you have a factor of 2. And this comes because the term which you neglect when you derive Larmor's theorem is negligible if the situation is that you have electrons and charges forming magnetic moments.

But if you have a free electron, the neglected term is exactly $1/2$ of the dominant term. And that is why the cyclotron frequency is twice about Larmor frequency. So never confuse the cyclotron and the Larmor frequency. And the factor of 2 is not related to a G factor of the electron or such. It's really the difference between the physics of a free charge and the physics of a magnetic moment. Any questions?

OK, well, then we are finished for today. A reminder, no class on Wednesday, but we have class on Friday in the different classroom.