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PROFESSOR:

Good afternoon. Welcome to April fool's day. But we'll be serious about atomic physics. I hope you enjoyed your spring break. At least I did. It's just a little bit change of pace. So let me remind you what we have been discussing. We have been talking about interactions between neutral atoms. And that can be trivial and very profound. Trivial because you've heard about van der Waal's forces. Polarization forces between neutral atoms. And if you describe them phenomenologically, you just have $1/r^6$ or whatever power of an interaction.

On the other hand, if you take the position that everything which happens between neutral objects has to be mediated, has to be created by the photon field, then it becomes a question of the vacuum, the vacuum which surrounds the two neutral atoms. And I've tried in this chapter to show you both sides, to similar or the same physics from two different perspectives.

We talked mainly about the van der Waal's interaction between two neutral atoms. I don't want one to-- we pretty much finished it, so I just give you a bird's view. I don't really repeat a lot of things. Because today we want to talk about the forces which happen between neutral atoms in metal plates and then two metal plates, which is the Casimir force.

But just to remind you that we showed that the forces between neutral objects are actually created by quantum fluctuations. Classical objects would not create fluctuating dipole moments, would therefore not create-- would not give rise to any force between neutral atoms.

And I showed you that there are two aspects where quantum physics comes in,

namely through the fluctuations or the zero point motion of the atoms as oscillators. And we even had a circuit model for that. But then I also said that, well, assume that you have two neutral atoms and you have the vacuum fluctuations of the field. This vacuum fluctuation is driving coordinated synchronized dipole moments between the two atoms, and then the two atoms can attract each other.

So we have seen the dual role which quantum physics plays here in terms of quantum fluctuations of the atomic oscillators and quantum fluctuation of the electromagnetic field, which is another harmonic oscillator. And at least when I use very simple pictures, it seemed that the $1/r^6$ potential, the standard van der Waal's potential at short range, comes from the vacuum-- from the fluctuations of the oscillator. I didn't do the derivation, but the simplest derivation for the long range Casimir Polder retarded potential, $1/r^7$, focused on the fluctuations of the electromagnetic field.

So these were some preliminaries. Then I thought I want to show you the power of diagrams by showing to you that there are two different kinds of diagrams dominant at short range and at long range. And with that we realize diagrammatically that, because of the nature of virtual photons which are exchanged by the two neutral atoms, there is a change in power law. The power-- the inverse power of the distance between the two neutral atoms is one higher at long range. And long range meant-- that also became clear diagrammatically. Long range means the distance is longer than the wavelengths of the resonant radiation for the atom. Any questions about that?

So the goal for the first hour today is to go from two atoms to many, many atoms, which then form a metal plate, and make the transition from forces between neutral atoms to forces between two metal plates, which is the famous Casimir force. And I want you to sort of hold the thought about what is really the quantum aspect here? Is it the vacuum? Is it the zero point fluctuations of the electromagnetic field? Or are the atoms the zero point fluctuations of the atomic oscillators responsible for those forces? So we'll come back to it after we have discussed the Casimir interaction between two neutral plates.

So maybe one highlight for you today is that maybe in an hour you have should at least have an expert opinion whether zero point fluctuations of the electromagnetic field are real or just a convenient way to describe some physics. But let's get there.

So the way how I want to make the transition from two neutral atoms to two metal plates is by reminding you that the potential was proportional at long range to $1/R^6$, at short range at $1/R^6$. And if you had the model that the long range interaction comes, because the vacuum fluctuations drive dipole moments in the two atoms, then we know that the potential is proportional to the polarizability of atom one times the polarizability of atom two.

So let's now extend that to the interaction between an atom and a wall. Well since I don't know-- at least not yet-- how to describe the wall. I take the position that, well, we have an atom come close to a wall, but now I extend the wall to a sphere of radius z .

And it's not obvious that if an atom interacts with the wall, if the wall were flat, it is of course this part of the wall which is most important. So we should actually get a quantitative or semi quantitative result out of that.

But now I can use a nice result from electrostatics, which I also discuss in 8421, in the [? MO1 ?] course, namely when we have a conducting sphere of radius c . Do you know what the polarizability of a sphere is? You must have also had this problem in classical electrodynamics. You apply an electric field. You calculate with the boundary condition of the sphere some spherical harmonic function. You calculate what the dipole moment is, and you find the proportionality constant.

AUDIENCE: It's like z^2 or [INAUDIBLE].

PROFESSOR: Yes. The one thing I want is that it's proportional to z^3 . It's the volume of the sphere. And the reason why I discuss that in atomic physics is when we talk about the dipole moment at the polarizability of the hydrogen atom, we find that the polarizability of the hydrogen atom is the pole radius cube. So it seems that when it comes to electric fields and polarizability, an atom, a hydrogen atom behaves

exactly like a conducting sphere, and the size of the sphere is now the size of the atom.

But that means now that the long range potential will now be proportional the polarizability of the atom. And we lose three powers in the power law, because we have three powers of z in the polarizability of the sphere. OK, if that worked so well once, we can now use that at least as a bridge to discuss what happens when we have two metal walls.

Well, we can now say we have two spheres, and the polarizability of each sphere is proportional to z cubed. Well when we have two walls, we want to use the potential per area. So what we had is we had 1 over R to the 7 . We lose three powers of R or z , because of one sphere. Another three powers of the other sphere, so it's 1 over z . But when we normalize by area, it becomes 1 over z squared. And now we find that the potential is 1 over c cubed. And at least with it I have-- I wouldn't say derived, but motivated for you the famous Casimir potential, which is 1 over c cubed. And I want to give you an exact derivation of the Casimir potential in the next few minutes. But before I do that-- yes?

AUDIENCE: Why did you choose the spheres to be the same size as the distance between them. That seems kind of--

PROFESSOR: It's all just a trick.

[LAUGHTER]

You know what would happen is-- let me just go back here. If I would choose this sphere smaller, I would not have given the atom the whole exposure. I have to make it at least larger than c to make sure that the curvature of the sphere doesn't dominate and what the atom sees is it sees there is a wall. But then you can see when I make this sphere larger than c , then I delocalize the dipole moment. And the bigger sphere has a big polarizability, but the atom which is so close to the sphere doesn't care about it. So you would say the right choice is just when you think either picture gets corrupted. But that's often what it is.

So if you think about it, it's the only choice. Both other choices would lead to an unphysical result, and you would immediately realize why. In one case it's the curvature, and the other case you're not really sampling the dipole of the sphere because you want to do a dipole field, and if you have a dipole of size z , the field with the dipole starts to be an approximation if you go distance z away. Other questions? Yes?

AUDIENCE: Can you explain where the first term [INAUDIBLE] z to the [INAUDIBLE] come from?

PROFESSOR: We had it. Last class we had the result that the Casimir-- that the van der Waal's-- the retarded van der Waal's potential is 1 over R to the 7 . But with the physical picture in mind that the van der Waal's potential at long range, which is 1 over R to the 7 comes because a vacuum fluctuation drive to dipoles, and the two dipoles attract each other. When each dipole is α 1 times a vacuum field, the other one is α 2 times a vacuum field. But now for this sphere, α is z cubed. So therefore three powers in the denominator cancel, and I get z to the 4 .

So you can say the α 2 , I absorbed-- I used the kind of result that α 2 is z cubed, and then the α 2 cancelled with three powers of the distance in the denominator. Other questions?

I have a question for you now. If you have two metal plates, what about short range and long range? We expect something under the rock. If we assume a metal plate, what is the vessel-- we had the distinction between long range and short range, which came from the resonant radiation from the resonant excitation of the atom. And when we are closer than the wave lengths, we're in short range, if you're further than the wavelengths of the resonant transition, we're in long range.

Between an atom and a plate, well, we still have the resonance of the atom. But what happens when we have two plates? There's clearly no resonance, so therefore there is only one power law, 1 over z cubed.

But I want you to sort of think about it for a moment whether this is now-- whether everything is long range or everything is short range. In other words, you say

effective resonant radiation, you say effective wavelengths of the plate λ equals 0 or λ equals infinity. Any opinion?

AUDIENCE: [INAUDIBLE] ability, so it must be infinite [INAUDIBLE]. Take the two metal plates, the vacuum energy between those plates, and compare it to vacuum energy at the same volume free space and subtract the [INAUDIBLE] energy difference. And high frequency components cancel out, so you have the--

PROFESSOR: OK, we'll actually do exactly what you suggest, to compare vacuum energies, in the next few minutes. It's not quite obvious, but I want to tell you now we have made-- by assuming the metallic boundary condition, by assuming that we have idealized metal plates which fulfill the boundary condition that the electric field is 0 at the surface. The parallel component of the [INAUDIBLE] at the surface. We've made an assumption here, and the assumption is actually that the resonant wavelengths goes to 0, and resonance frequency goes to infinity.

And the power law we obtain is actually the long distance power law so to speak, and for two metal plates, there is no short distance. I'm not sure if I can give you a simple explanation because the boundary condition is so idealized that you don't recognize, you don't smell any atomic resonance anymore. But maybe the one thing I would say is that you say the boundary condition, metallic boundary condition for infinitely high frequency. And sometimes that would also mean that the plasma frequency of the metal is pushed to infinity. We come back to that in the next chapter. Any questions?

AUDIENCE: [INAUDIBLE]?

AUDIENCE: Excellent question. I will go for you now through-- its mainly classical e and m . This is why I use pre-written slides. We sum up over all modes with a boundary condition, with a metallic boundary condition. If you have in general a dielectric or semiconductor which is not an ideal metal, has different boundary conditions, the Casimir force will change.

And this is actually a pretty outstanding difficult problem where people have really

chewed on hard. I think Lifschitz has some famous solution on it. What is the Casimir force for an arbitrary dielectric constant of the surface material? But absolutely yes. If you don't have a metallic boundary condition, your Casimir force changes.

I think this is sort of an interesting point, and it may trigger immediately discussions what does it really mean. But let me go now through school the derivation where we have metallic boundary condition, and this actually leads to a quantitative derivation of the Casimir force. Quantitative in the way that we get-- we're not just getting the power law. We get the prefactor. We everything 100% of this derivation. Collin?

AUDIENCE: So what is the mechanism for the Casimir force changing? Because if think of it in the picture of two metal plates, the lower density and modes inside and then do outside. Even with the arbitrary boundary condition, I [INAUDIBLE]. But what is the physical mechanism for changing the form?

PROFESSOR: To the best of my knowledge, it is the density of modes. But it's very subtle. I suggest, since not everybody is probably familiar with the way how I will count all the modes in the metal plate, let me just do that, and then we should-- and I hope then it's obvious, or we should then discuss the boundary conditions being modified. And you have a question?

AUDIENCE: Yes, but maybe we can discuss it after this.

PROFESSOR: So the idea is the following, if the Casimir force is only due to zero point fluctuations of electromagnetic field-- let me say up front this is a valid assumption. I get a valid result. But I tell you in half an hour that you can take another perspective and say this is just a holistic description. There are other derivations of the Casimir force which do not use the concept of zero point energy at all.

So in other words, I'm presenting you one presentation of the problem, one solution of the problem, but there's a whole different formulation. But here I assume-- mean I can easily modulate. OK, well here we have two metal plates. We want to understand if there is potential energy and interaction between the two metal plates.

And let's just calculate what is the total energy of the system. And if I have two metal plates, which are two mathematical boundary conditions, all I have to calculate is the ground state of the electromagnetic field in between. And that means I have to find all of the modes, and each mode contribute $\hbar \omega$ over 2.

So this is our agenda now. We say the metal plates are not real objects. They're mathematical boundary conditions. Looks like a good assumption. And now based on those boundary conditions we calculate the zero point energy of the world, at least of the electromagnetic world. And then we ask if the plates are a little bit closer, does the total zero point energy change? And when it has changed that means there is a force between the atoms because of potential changes as a function of distance. So this is the agenda I want to execute now. Any questions about what we want to do?

As you know when you write out modes in e and m and you want to do it exactly the many, many indices. But this is a result of which could actually come straight out of Jackson. For your convenience, this is also discussed-- a wonderful discussion by Serge Haroche. In some of these summer school notes from the '70s or '80s. I've posted those on the web. So if there is a mathematical detail you want to look up, this is posted on the web.

So anyway, we have two metal plates with a distance l . And we have metallic boundary conditions. That means that when we calculate the modes, we get TE modes and TM modes. TE modes means there is no longitudinal electric field. TM modes mean there is no longitudinal magnetic field. And the frequency of each mode is discrete in the z direction. In the z direction we have standing waves, which are labelled with an integer value of m , whereas parallel to the plate the modes can have an arbitrary transverse wave vector k .

And when we look at the dispersion relation, the frequency of those modes is in quadrature. You can see the component $k^2 c^2$ of the transverse propagation plus the component of the standing wave. So for each m , for each integer value, we have a TE and TM mode, with the exception of m equals 0. For m

equals 0 we don't have a TE mode. We only have a TM mode. So that's just fresh out of Jackson.

But it's very important now-- and I want to spend a little bit more time with it so it's just a purely classic result. What is the density of states? Well, let's assume we have a given m . So we have a standing wave perpendicular to the capacitor plates, and therefore now the density of modes only comes-- the density of modes only comes from the transverse. Let me just get my toolbar back. So it is from this k squared. And you know that the number of modes is proportional to the volume in k space. And the important part is that I can write this as dk squared. But k squared is $d\omega$ squared. And this gives me $\omega d\omega$.

So for each n value, for each discrete mode perpendicular to the plates, we have now a density of modes which goes as $\omega d\omega$. Before I show you the mathematical expression, let me just sort of plot that, what that means. The density of modes was-- the number of modes was $\omega d\omega$. That means the density of modes was proportional with ω .

That means for m equals 0 we started ω equals 0, and we get a straight line. For m equals 1-- m equals 1 shown in green, and m equals 2 shown in purple. The frequency ω -- we have ω squared is sick the quadrature sum k squared and m squared. So therefore if you want to ask what is the density for modes for m equals 1 or m equals 2, you have to start at the lowest frequency for m equals 1, at the lowest frequency for m equals 2. But then as we just derived, the density of modes is proportional to ω .

So what you we to do in the end, we want to sum those modes all up. And what we will observe is that here we have a line with slope 1. Here we have to sum two lines with slope 1. Here we have to sum three lines with slope 1. So progressively, the slope gets higher and higher.

But the slope, which increases linearly with the coordinate axis. If it were continuously, would just be a parabola. So in other words, if I would [INAUDIBLE] the density of states which comes from the exact mode spectrum, if I would

[INAUDIBLE], it would be parabola. And parabolic density of state, the number of modes is $\omega^2 d\omega$. This is three dimensional space.

So in other words, what we observe is what exactly the effect of the boundary condition is. Instead of getting this smooth parabola ω^2 , we have sort of broken up the parabola into piecewise linear pieces. And the fact that we have a boundary condition, that we plates in free space, and not just the space, is actually the difference between the two. So it is the difference between the two, which leads to the fact that a boundary condition modifies the vacuum energy. Questions?

OK with that the rest is mathematics, but I want to show you the steps because it is ingenious mathematics. It makes a few approximations. It creates a few infinities and gets rid of it. It's actually rather wild, but amazing that, in the end, we have a simple exact result.

OK so the most important thing is we have a density of states. And the density of states for one m is proportional to ω , but now we have to multiply-- when we want to know what is the density of state in a given ω , we have to multiply with the integer number of discrete modes which are available at this ω . So this is because there is only one m equals 0 mode. We have a one here. And then for m equals 1 and higher, we have two [INAUDIBLE] between TE and TM. So this is just counting the modes.

And this is now a mathematical trick that, instead of counting the modes-- it's an integer-- we can sum up heavy side step functions and sum over all m. This is just an [? exact v, ?] right? But this is something which helps us in the next step to do an integral.

So now so far we haven't made any approximation. We've just reminded ourselves what is the exact result of classical e and m. OK, but now we want to calculate the zero point. So now it's getting interesting, because we take the classical result, and take it into quantum physics.

The first step is very innocent. We say each mode contributes $\hbar\omega$ over 2.

And then we integrate with the density of states. And the next line is also an exact result, just using the density of modes from the previous page. But of course, now we are asking for trouble. We are integrating with $\omega^2 d\omega$, and there is a UV catastrophe.

Now I can say I can put on my mathematical head and say let's add a convergent term, something which exponentially decreases with ω , which is just cutting off the high frequency part. And I hope that in the end-- and this will come out too-- that I get a result which is independent of the cutoff. So by choosing this cut off parameter λ to zero, I have-- if this is zero, I have no cut off. So maybe I can introduce a cut off, and then at the end let λ go to zero. And if this gives me a physical result, I've sort of mathematically tricked around.

But I also like the physical justification for that. I mean you know that a metal plate has not metallic boundary conditions at infinitely high frequency. If you go deep to the x rays and gamma rays, it becomes transparent. So therefore, we would argue that, once we go to very, very high frequencies, the modes do not feel that there is a boundary condition, and therefore we may be allowed to do some physical cut off.

OK so we do this cut off by-- so now we have to find the zero point energy. The zero point energy has a sum over m . And I call each integral $I_{sub n}$. And this integral is the ω^2 from the density of states, but now multiplied with a convergence term. Whether it's mathematical or physical.

And by introducing this cut off, I can now exactly solve the integral. The ω^2 can be taken out of the integral by taking the second derivative. And this is just an exponential function which can be integrated. And therefore we have a mathematically exact result. So we are now able to do the integral because we have avoided the divergence.

OK. Now we come back and say we are really interested if the cutoff is pushed to higher and higher frequency. And this means λ equals 0. So therefore, we want to take this expression and expand it around λ equals 0. Well, you could

say that's physically very well motivated. But if I take this function, which is $\lambda e^{-\lambda x}$, and expand it, I get divergence is $1/x^2$.

So you would say how can you expand it. Let's see what happens. Let's just do it. Let's say we are interested around λ or x equals 0. We have some divergencies now, but we have to find a way to deal with it. So we get rid of some of the divergencies in the following way. By saying, well, if you simply calculate the zero point energy between those two capacitor plates, and then when, let's say, if there is-- if the plates come closer and there is less zero point energy, we would see the plates attract each other.

But what happens is the plates are not moving, and there is no world behind them. The world is sort of as it is. And that means we have to sort of now look more carefully at the zero point energy of our capacitor plates where l is small. And the rest of the world. So we are now representing the world as a big capacitor of size l_0 . And then our plate is just moving within that, because we are not assuming that then the two capacitor plates attract each other and move towards each other that the size of the universe is changing. So we want to move a boundary condition within the universe.

So therefore, the correct expression for the zero point energy is not the zero point energy of capacitor l . It's the zero point energy of a capacitor size l plus of a capacitor l_0 minus l . And so we take the result and simply add the expressions for l and l_0 minus l .

And what is now a nice is that the first highly divergent term with $1/\lambda^4$ reminder that we want to let λ go to zero. It becomes independent of l . So we would say, well, if there is an infinity, it doesn't change with l . It doesn't provide a potential. It's just one of those infinities which is constant, which is not affecting the physics. Also here, this is independent of l . And then the next term gives us the famous Casimir potential, $1/l^3$, and it is independent of λ . So we have the result which was independent of the cut off we introduced

earlier.

So therefore, we have derived, with a little bit of hand waving to navigate about infinities, that the potential between two metal plates is $1/d^3$. d squared was the area, the size of the capacitor plates. And this is the exact prefactor.

Or if you want to figure out what is the pressure of the vacuum. Pressure is, of course, force per unit area. So you take the derivative of the potential that's the force, that's the force per unit area, and this is now the vacuum pressure. Sort of funny now you can put in the unit. It's 10^{-5} millibar at a distance of one micrometer.

So anyway this is, I think, the classic deviation of the Casimir potential by, I would say, exactly summing up the zero point energy within a certain boundary condition. Good.

Maybe to address Collin's question. What would happen if the metal plates were maybe semiconductors or had a different dielectrical constant. The mode spectrum would change.

AUDIENCE: Like if you had mixed boundary conditions. You don't have these nicely spaced--

PROFESSOR: I don't know how to calculate. That's a hard problem, and a lot of people have really worked very, very hard on it. It seems for idealized boundary condition you can do it easily, but it has been clearly a challenge to mathematical physics. And I'm not even sure if it has been solved in all generality or only in special examples to extend that to every type of surfaces.

But we can, for instance, assume if you have a conductor, which is not-- conductor which has not infinite conductance, 0 resistivity, then the electric field can penetrate a little bit into the conductor. And if the electric field penetrates into it, it changes the frequency of the mode. So it's clear that the boundary conditions have an effect here.

But let's now have some discussion. We had focused before on the force between

two atoms. And if we use a microscopic model for the metal plates, it should be, it must be possible to obtain the Casimir potential by just summing up all the forces between atoms. Because ultimately if you have no metal plates, you don't have any force. Where s-- in the last few minutes, we focused on the derivation which was simply summing up the zero point energies.

And there are two papers which I've posted on the wiki which argue that both results are equivalent. So I would really recommend to you-- he expresses it much better than I can. Read the introduction and the conclusion from Bob Jaffe's paper. He clearly says in the introduction that there are a lot of books and references by famous people who have said the Casimir force is the manifestation of the zero point energy. The fact that people have observed the Casimir force means that $\hbar \omega$ over 2 is real.

Those authors take the opposite approach. They say this is just heuristic. It's convenient. You can get the correct result. But there is an equivalent derivation which gives exactly the same result, which focuses only on pair wise interactions between atoms, and it does not include any-- it doesn't make any reference to the zero point energy of the electromagnetic field.

So therefore, one, I would agree with those authors that the conclusion is there is absolutely no observational evidence for the zero point energy, that the zero point energy is sort of not just convenient counting, but that it is real energy. And the Casimir force, of course, also it has often been quoted as a counter example that this is a direct observation, is not-- should not be regarded as such.

Now people at MIT probably know Bob Jaffe. He's a high energy physicist. And why is he interested in atomic physics? Why is he interested in the Casimir force? Well, the question of zero point energies is very, very important in these days for dark energy. I mean, you know that 80% of the energy of the universe is dark energy. Then there's dark matter, and then there is baryonic matter. But this dark energy is really mysterious.

And one candidate for dark energy is the dark energy simply comes from the zero

point energy. So therefore, people are now really interested is the dark energy real. And if it is real, does it have a gravitational effect? Does it change the metric of space? And can it accelerate the expansion of the universe?

Of course, you have a problem if you calculate the zero point energy, you don't introduce a cut off. You have a spectacular uv divergence. You can maybe now concatenate the divergence by saying, OK, eventually if the wavelengths become shorter and shorter and shorter, there is a cut off.

And if you have no idea at all, you would introduce a cut off at the Planck lengths, because beyond the Planck lengths all physical descriptions break down. But if you cut it off at the Planck lengths, you find that the zero point energy of all modes is larger than the observed dark energy in the universe for acceleration of the cosmic expansion by 124 orders of magnitude.

And if you think you want to introduce a cut off which is maybe the classical electron radius. The classical electron radius is the radius of a charged sphere for which the electrostatic energy would just be the rest energy of the electron. Well, then you're off by 43 orders of magnitude. So some people have called those discrepancies, which are completely unresolved as of now, the biggest failure ever of theoretical physics.

So anyway, that's why there is renewed interest in the Casimir force. And at least in electromagnetics, we have a very, very simple model. And we have experiments which measure the Casimir force, but, as I've said, it cannot be regarded as a direct observation of the zero point energy.

OK, let me now come to a question we started discussing earlier, namely if we-- here we have atoms. Here we have a metalized boundary condition. So maybe let's just talk for one minute about it. What kind of assumptions do we make about atoms when we replace them by a metallic boundary condition? From the whole derivation of the Casimir force it's not obvious that there are no atomic properties which have entered the calculation of the Casimir potential.

What Bob Jaffe argues in this paper-- and again, it's wonderful reading. I really recommend Just if you want to enjoy some physics, open up the document and read the first and the last page. The Casimir force from an idealized boundary condition is completely independent of atomic properties.

So therefore, it must be a limit where-- well, we talked about the resonant wavelengths, but we can also talk about the fine structure constant, because the fine structure constant in a hydrogenic model provides what that resonant transitions in hydrogen are. So you can argue if something is completely independent of atomic properties, somehow by sweeping things under the rug we must have set α to either zero or infinite. Because if it is set to any value, it would correspond to some version of the hydrogen atom which has a resonant radiation.

So the question is, is α set to zero or is α set to infinity when you derive the Casimir force? And I think this is now easy to discuss, because the fine structure constant α is e^2 over $\hbar c$. If you set α to zero, you do it by setting the charge of the atoms to zero, and then you have no interaction at all.

So therefore, if you want to have the Casimir force in a limit which is independent of atomic properties, the α equals zero limit gives no force. So therefore, the equivalence this picture, summing up over the electrons and obtaining the Casimir force for the metallic boundary condition, this equivalence happens when you do this derivation and set the fine structure constant to infinity.

How big is the fine structure constant? $1/137$. In some expansions of α and α^2 , it's actually a smaller parameter. And now I'm telling you when you have two metal plates and measure the Casimir force, this corresponds to the infinite α limit.

So this is now getting complicated and I can refer here only to Bob Jaffe's paper. But he shows that when you go to distances of 0.5 micron that people have measured the Casimir force. You are already the α equals infinity limit. If you are-- I wish I could read my handwriting. I think that's 10^{-5} . If α is larger than 10^{-5} . So therefore, α equals $1/137$ is for that purpose already

very, very well in the large alpha limit.

But I just hate this result here. This is rather complicated. Bob Jaffe says that to get the Casimir potential by summing up over atoms is actually a quite challenging mathematical exercise. Questions? Yes?

AUDIENCE: If these calculations of summing over atoms has been [INAUDIBLE], then what is it-- we can just talk about the plates being dielectric from that atom [INAUDIBLE]?

PROFESSOR: There are solutions for dielectric plates. And sometimes when people do experiments, the best surfaces you can get is a silicon surface. Because they want you to do qualitative measurements, they're discussing corrections due to the dielectric constant.

I know the problem has been worked out for dielectric surfaces. I assume not by doing the transition from summing up atom by atom, but by rather looking at the boundary condition for electromagnetic fields in the presence of a dielectric. But I don't know more details about that. Yes?

AUDIENCE: This question may not make sense because I haven't thought [INAUDIBLE]. But again, when we consider two plates, if you have room for two atoms, we get the two $1/R^6$, $1/R^7$ cases. One of which is the retarded Casimir potential, but for two plates-- again, we conclude that it was more like the short range--

PROFESSOR: Well, long range. We are always retarded.

AUDIENCE: OK, but then I'm a bit confused. Because then if you say alpha goes to infinity, don't you have an \hbar or c to zero? Oh wait. That makes sense.

PROFESSOR: Alpha infinity is-- we know that this is the vastness of the electron. The [? Whitbeck ?] constant is alpha squared times smaller. And the [? Whitbeck ?] constant-- well, a quarter of the [? Whitbeck ?] constant is alignment alpha radiation. So anyway, the [? Whitbeck ?] constant is a scale factor in the transition frequencies of the hydrogen atom.

And so therefore, it is consistent that, if you make alpha large, we actually get-- we push the frequency of the atomic transition to very, very high frequencies. And that would mean the resonant wavelengths goes to zero. And therefore any finite distance between the capacitor plate is, in the long range, is longer than the wavelengths of the transition.

AUDIENCE: I think equivalently you could say if the speed of light is really slow, you always will have [INAUDIBLE].

PROFESSOR: Yeah, sure. You can do the transition in many ways by tuning the speed of light, or tuning the charge, but it all has the same implication. Other questions? Lindsay?

AUDIENCE: Is there any other observable reason why we should constantly be thinking about Casimir forces and dark energy? Like experimenting [INAUDIBLE].

PROFESSOR: As far as I know, this question about dark energy and whether it is related or not to zero point energy of the field is completely open. I think if you had an idea how to do a decisive-- if any of you had an idea how to make a decisive experiment measuring whether zero point energies are real or not, that would have a big impact. As far as I know, it's completely open. Nobody has really good idea. So some people feel, mathematically, the zero point energy has the right, correct physically characteristics.

Zero point energy could explain the extra term in Einstein's field equation, which is often called the cosmological constant term. But the quantity, it's so many orders of magnitude off that people don't even know if that is correct or not. One could also say, well, maybe zero point energies do not give rise at all to any gravitation potential. And therefore maybe if energy doesn't give rise to gravitational potential, maybe the energy is not real. And what we find, what is responsible for the expansion of the universe is gravity-- so Einstein's-- what's the constant called again? I just mentioned it.

AUDIENCE: Cosmological.

PROFESSOR: Cosmological. So that is just the cosmological constant, but the cosmological constant reflects something else in the universe and is not related at all to zero point energy. To the best of my knowledge, these are the possibilities.

OK, so let's go back to atoms. Our next big chapter 12 is on resonant scattering. Just that you see the structure of the course. I wanted to introduce sort of some mysteries and subtleties of electromagnetic interactions to you by using diagrams, by using an exact formulation of QED. And then I gave you one example where we could use those diagrams, and these were van der Waal's forces.

So we got a little bit side tracked with the vacuum, with van der Waal's and Casimir forces, but now we want to go back to the diagrams. And there is one aspect of the diagrams which is sort of fascinating, and this is where we really need a treatment which goes beyond perturbation theory. And this is when we have resonant radiation. When resonant light interacts with atoms. Because, as I'm going to show you, we have then in any perturbative expression zero in the energy denominator, and we would have divergence.

So I want to show you now, partially also for the beauty of the physical theory, how can we deal-- how can we fix those divergencies? How can we fix those infinities we encounter in perturbation theory? The answer will be we get rid of these infinities by summing up an infinite number of diagrams.

So I'm going to show you now-- it actually sounds so much more harder than it is. How, with rather modest mathematical effort, we can perform an infinite sum over diagrams. And I want to show you what those diagram are. And the result of that is that the divergencies disappear, and we have a valid description of what happens when atoms encounter resonant radiation.

So I'm following here, almost religiously, the atom photon interaction book. So this in production is a summary of those pages. So when we discussed diagrams, we wanted to figure out what is the amplitude for a system to go from an initial to a final state. And those were the matrix elements of the t matrix, the transition matrix.

And if we do it in second order-- well, remember the structure is in second order. We sort of emit a photon, virtual or real. We have another photon. This gives us second order. And in between the system propagates with its eigen energy.

So this was the structure of the diagrams we had. And this term here, let's expand it in one wave functions of the unperturbed Hamiltonian. And then we obtain this following structure. So this is just rewriting in a very suggestive way what we had discussed before.

And the case I focus on now is the interaction of atoms with resonant light, because this is when we have divergencies. If we have state a, state b. We have a laser and we scatter light. If the frequency is approximately resonant, then those perturbative results have a divergency. So for resonant light scattering, the typical matrix element we are interested in, we start and end of course in the ground state. We scatter photon. We go up to an excited state and go back to the ground state.

But we started initially with a photon of wave vector k and polarization ϵ . And this may be different for the scattered light. The interaction happens by the interaction operator, which, for the purpose of this discussion, can either be the $A \cdot p$ term or the $d \cdot E$ term. We've discussed about the special role of the a^2 term. We discussed it earlier. I don't want to include it in the discussion right now.

And then you would say-- just want to make sure they do not confuse. In second order perturbation theory, you always have an energy denominator. But as I also explained to you, the energy denominator comes from the free propagator. You had a dot propagate and a dot-- and what is the time evolution of this free propagation when you integrate it with time? This gave rise to the energy denominator. Just a side remark.

So you should be very familiar that we have an energy denominator. But now this energy denominator-- the initial energy is the initial atomic state plus the photon. And then if you want to sum over all states, I can just put in here the operator H_0 . But you may immediately think about it, the only state which really matters is the

state b . And what matters is only when we go through intermediate state b , then H_0 is E_b , and then we have a divergence because the denominator is 0.

So let's see how we can fix it. But let me first give you the perturbative result, which you have seen many, many times. It is the product of two matrix elements. We have an intermediate state. The excited state without photon. We started with a photon, and we end up with a photon. And here we have our dipole or $A \cdot p$ term. And our energy denominator diverges when the frequency of the photon approaches the resonance frequency.

Well, how can we fix it? You know that in many cases you're just adding an imaginary part to it, which reflects that, due to the coupling with the vacuum of the electromagnetic field, the excited state is broadened. And it gives its energy an imaginary part. This of course is very phenomenological, but at least technically it avoids the divergence.

Now what I want to show you, in the next 15 minutes and we continue on Wednesday, is how I can get this result of a very rigorous approach. I want to use the diagrammatic approach. I want to sum an infinite number of diagrams. And will tell you what are the approximations to get this. For instance, as you know, when you have an imaginary part, when you have just an imaginary part of the energy, the physics of it is exponential decay. Well I hope, not today but on Wednesday, you realize what assumptions lead to exponential decay, and that, in principle, non exponential decay is rather the standard than the exception. But the effects are small.

But I want to show you. So in some sense I want to give-- we have to go through a half an hour of work to derive this result. So you'd say, what's about it? You just get the phenomenological result. Well, we have put it on a rigorous basis. And number two is we do understand what assumptions really go into this phenomenological result.

So the way how we will actually obtain this result is that in-- the second order diagram is that you start in state a , you go in state b , and you're back in state a , and

you exchange photons. So the way how we-- the physical picture we want to use now is after the photon is absorbed right here, the system is not propagating with the Hamiltonian H_0 . It will propagate with the Hamiltonian H .

If we do that, then we avoid the divergence. We get a result which can be applied toward-- the formalism can be applied to many situations. And in one limiting case, we obtain that. So the physics behind it is the following. The divergence which we encounter in resonant physics is only a divergence because we assume the atom in state b is naked. It's an excited state of an isolated atom. But in reality, the state b interacts by virtual photon emission, interacts with all the modes of the vacuum.

And once we throw that in by saying that between two-- between absorbing and emitting a photon, the state b propagates not with a free Hamiltonian, but with the total Hamiltonian, we avoid the divergence. But this is no longer a perturbative result. And let me illustrate that to you. If you take the energy denominator, $\omega - \omega_0$, and we add the imaginary part to it, I can Taylor expand it, assuming the imaginary part is small.

Well, that's sort of a trivial expansion, but now you should know that γ , the rate of spontaneous emission, the simplest expression is obtained in [INAUDIBLE] is called γ in the second order in the interaction Hamiltonian. Spontaneous emission is proportional to the atomic dipole matrix element squared. So it's second order in the Hamiltonian. But that means that this harmless addition of an imaginary part means that if you look at the Taylor expansion, that we are using now infinite orders. Infinite orders in this expansion.

So the phenomenological addition of [INAUDIBLE] γ to the excited level b is actually highly nontrivial, because it's corresponds technically or physically to a non perturbative result which involves infinite order in the interaction matrix element. Question about it?

So this is more setting up the agenda. I wanted to remind you of the perturbative result. I want to sort of show you how we can fix it. I want to show you what it means. But with that motivation I want to now back to our diagrammatic approach,

our formal solution, our exact formal solution of the time evolution of a [INAUDIBLE] system, and show you one how I can implement this agenda in particular by not allowing the atom to propagate with a naked Hamiltonian H_0 , but sort of staying dressed with photons and propagating with the Hamiltonian H . Questions about that?

OK, so the derivation is now focusing on two chapters in API. I need one result which I hope you remember from about a week and a half or two weeks ago. We formulated the time evolution of the system using the time propagation operator U . And we found an exact expression. I'm dropping all of the arguments here, t and t' . You can look them up either in the book or in the previous lecture notes. We found that the propagator in the interaction representation is the unperturbed operator U_0 .

Now I need, actually, the terms. It was a time integral over some time t_1 , which involved U_0 , V , and U . And I introduced it to you as you could plug that into whatever is the equivalent of Schrodinger's equation for the time evolution operator, and you really find this is a solution for U . But well, it's not really a solution, because it's a solution of U expressed in terms of U .

But then I showed you that it allows an iterative approach, and this is why we did it. If you have a first order result, the zero order result is U_0 . When you plug the zero order result in here, you get the first order correction. When you put the first order result here, you get the second order correction. So you get a recursive formula which solves the problem.

And this is now our starting point. This is an exact formulation of QED. And this is now our starting point to address the problem of divergencies. We want to describe what many of you do in the lab, namely have resonant laser light interact with atoms. And I want to show you how can you describe it with QED with the formalism we've introduced.

OK, now for this chapter we want to have one big simplification. Things aren't only getting complicated. They're getting simpler. If you have a time integral which is a

convolution, and you can often simplify it by Fourier transforming it, because the Fourier transform of a convolution is simply the product of the two Fourier transforms.

There is one glitch. So one subtlety I want to feed in a moment. But let's just assume for a moment, we go from time before the Fourier transform to frequency or energy. So G of E is the Fourier transform of-- after filling a few corrections of U . That would mean, if that were true, the equation above would now be an equation where the integral, V sort of constant, instead of a convolution of U_0 and U , we get simply the product of the two Fourier transforms. That's much, much simpler. It's an algebraic equation instead of an integral equation. And remember, when we get into n th order, we get n integrals with some time ordering. We were able to do it, but it's mathematically more complicated.

The problem is that this is not a convolution. It would be a convolution if the time limits were infinite. So what we can do instead is we can define G of E . Not the Fourier transform of U . If we say it is the Fourier transform of k , which uses a step function, then if you insert now k in the formula above, you see that because of this step function we can now take the integral limits from minus infinity to plus infinity.

So therefore, when I say G , G is called the resolvent. It's really the mathematical quantity we have to work on now. And I wanted to tell you very simply hey, it's just a Fourier transform of the time evolution operator. Not quite. It's the Fourier transform of a time evolution operator with a theta function, just to make sure that formally we can extend the upper and lower integration limits to infinity. Or this is also a name-- what I'm really kind of discussing are Laplace transformations.

But I want to convey to you the physics. It would take many weeks to do the mathematical accurately, but you can get the idea and really profound feeling for how QED works and what we are doing with not so much mathematical rigor. So with that grain of salt we have Fourier transformed equation. We have the Fourier transform of the time evolution operator. But if you look at it mathematically more exact, it's a Laplace transform, and we are little bit navigating in the complex plane.

The Fourier transform does not exist for real values. We always have to add a small imaginary part to make things converge. But you've seen that in many other places, I assume.

OK, so the quantity which allows us to describe resonant interaction in a very straightforward way is called the resolvent. I can simply define the resolvent. All I gave you was some motivation. But if I had wanted, I could just say the resolvent is defined by this operator equation. z is the complex number. H is the Hamiltonian including interaction between atoms in electromagnetic field.

And actually if you would take that-- if you take this and plug it into the equation above-- so if you plug this into the equation, you find this expression solves the equation. So I just want to say you can forget about all the subtleties with Laplace transform and such. You can just say let me define some interesting object. But it is connected to what I said earlier, because this solves this equation.

And also I like to sort of show you the mathematical structure. If you write the time evolution operator as $e^{-iHt/\hbar}$, and now you Fourier transform it. If you Fourier transform it, you multiply it with $e^{i\omega t}$, and you integrate-- let me integrate from 0 to infinity. There are some issues with 0 or infinities. But if you simply solve this integral with time, you have sort of an exponent $e^{i(\omega - E/\hbar)t}$.

So if you Fourier transform it, what you obtain is nothing else than the energy minus $\hbar\omega$. And this looks very different. This looks very, very similar to the definition of the resolvent. The only mathematical thing I'm not fully discussing here is z real, or has it a small imaginary part? The small imaginary part-- actually, you have to add a small imaginary part here to make sure the Fourier transform exists. But you should clearly see how things are connected.

So OK, I'm focusing now on the resolvent. It is equivalent to the time evolution operator. It is a sort of Fourier transform of it. And we want to formally write down a diagrammatic solution for the resolvent. And with that I think I should stop now. We start on Wednesday? Any questions to that point? OK, then see you on Wednesday.

