PROFESSOR: Back to master equations an optical Bloch equations. I hope you remember that on Monday we derived, under very general conditions, how dissipation, relaxation comes about in a quantum system, namely because we have a quantum system which interacts with a reservoir. The total system evolves, as quantum mechanics tells us, with a unitary time evolution. When we derive life what happens with the small system, due to the coupling with a bigger system, there are now suddenly-- the time evolution of the density matrix has relaxation terms, and we derived that. And the special equation we are interested in for two level system interacting with the vacuum-- interacting with a vacuum for spontaneous emission are optical Bloch equations.

Today I want to discuss with you some characteristic solutions of the optical Bloch equations. But before I do that, let me first discuss with you the assumptions we made to derive a master equation. And I hope you remember the Born approximation and the Markov approximation.

The Born approximation says that the reservoir will never change. The reservoir is always, for instance, in the vacuum state. And the Markov approximation said that the correlations in the reservoir are delta function related. It can absorb photons instantaneously. The vacuum has no memory time. It's like a black hole. It sucks up everything. But it's always what it was initially, namely the vacuum.

Do you know examples where those assumptions are not fulfilled? I make it so natural. These are the most natural assumptions, but what are systems which cannot be described with this approach.

AUDIENCE: I was thinking maybe like an atom in a cavity. If you change the vacuum somehow.

PROFESSOR: An atom in a cavity. Very good. If an atom emits-- now it depends. We talk today about the master equation for photons and atoms in the cavity, but we will assume that the cavity is rapidly damped. If it's a high Q cavity, the atom emits a photon, but the cavity has memory. It stores the photon, and after one period of the vacuum Rabi oscillation, the photon goes back to the atom. So the vacuum has memory time. The vacuum has now a time constant, which is the same as the time constant for the atom, namely the period of the Rabi oscillation.

Big violation of the Markov approximation. We do not have the very short correlation time of the system. We just integrate over in the much longer time scale, during which we are interested in the dynamics of the system. Maybe another example. Actually, this example violates both the Markov approximation, also the Born approximation, because the vacuum has changed when it has a photon.

Do you know an example where maybe only the Born approximation is violated, namely the reservoir is changing due to interaction with the system? Well? Before we talked about the system in a big reservoir. What happens when you make the reservoir smaller? Well then when it absorbs energy. It will heat up and such. It will change. The Born approximation is actually, when it would come to energy transfer, the assumption that the reservoir has an infinite heat capacity. It can just take whatever the system wants to deliver without changing, for instance, it's temperature.

OK, so at least you know now there are systems for which the treatment has to be generalized, and you understand maybe better the nature of approximations we have made. [INAUDIBLE].

AUDIENCE: Just a question. For the last case which is where the Born approximation is not violated, is there any example of that?

PROFESSOR: Well for instance, spontaneous emission into the vacuum, but the speed of light is so slow that it takes forever for the photon to escape. The Markov approximation is more about the time scale for the reservoir to react. And the Born approximation is if the reservoir is really big enough to simply absorb everything without changing.

I don't know. You could maybe think about you have some reservoir, but it has very bad heat conduction or something. The transport is just very slow, and therefore you do not have the hierarchy of the two time scales. OK, any other questions about the master equation? Optical Bloch equations?

Well, then we can learn more about the characteristic features of the optical Bloch equations by looking at solutions. And I would like to discuss three aspects of solution. I want to discuss the spectrum and the intensity of light emitted by the atoms. Secondly, we want to talk about transient and steady state solutions. And finally, actually I put it in the chapter of optical Bloch equations, but I will use a cavity for that example, which will actually go beyond optical Bloch equations. It's also nice for you to see that there is more than the optical Bloch equation, but it's the same formalism.

And this is when we discuss the damping of the damped vacuum Rabi oscillations. I also picked this one example sort of as an appetizer, because I can introduce two concepts for you in this last example. One is the quantum Zeno effect. And the other one is the adiabatic elimination of coherences, which plays a major role in many, many theoretical treatments.

But let's start with the first part, namely the spectrum and intensity of emitted light. And I want to start off this discussion with a clicker question. Do everybody take a clicker out of the box? So the situation is that we have an atom which is excited by laser light of frequency omega L. The atom has frequency omega 0. I'm discussing the limit of very low intensity. And the question for you is what is the spectrum of the emitted radiation?

So in other words, we have our atom. We have the laser beam which comes here. And now the atom emits in all directions. And you collect and frequency analyze the light. And I want to give you four possibilities how the spectrum of the emitted light may look like. Let's assume that on the frequency axis this is the laser frequency. And this is the frequency of the atom.

And the first possibility is that we observed a spectrum which is centered at the atomic frequency where the line meets gamma. The second option is that we observe a delta function, a very sharp peak at the frequency omega 0. The third possibility is the we observe a sharp peak at the laser frequency. And the fourth option is that we observe radiation centered at the laser frequency, but with gamma. So I would ask you for your vote. A, B, C, or D.

Was there any questions about the choices? 40 for in spectra. The low intensity limit of light scattered emitted by an atom which is excited with a laser beam. We can assume close to resonance, but not on resonance, so we can distinguish between the laser frequency and the atomic frequency. Anybody had a chance? OK. OK, good. A, B, C, D.

Yes, let's discuss it. I want to draw your attention to one thing I said, namely low intensity. At low intensity, you can assume that there's only one photon scattered at a time, because the intensity-- we work in the limit of infinitesimal intensity.

So I want to as you now, why don't you consider energy conservation and think about the problem again? OK, maybe I'm indicating I'm not happy with the answer. Maybe an obvious question which sometimes people get confused about, what about Doppler shifts? We assume the atom is infinitely heavy and cannot move. Or we trap it in an ion trap. It's in the ground state of the ion trap, and no kinetic energy is exchanged. So don't get confused, and I think none of you got confused about Doppler shifts in kinetic energy here. We eliminate that. OK.

Well it's moving in the right direction. Now the two most frequent answers are those two. OK, I've asked you to take energy conservation really seriously. If you start with an atom in the ground state, and the monochromatic photon at the laser frequency, and you've done a wonderful job, which many labs can do. You've stabilized your laser with a precision of 1 hertz. And 1 Hertz, this is sort of what I do as a delta function.

So if you take energy conservation absolutely seriously, the photon, which can-- the atom is in the ground state after it has scattered light, so the energy of the atom has

not changed. And you can only fulfill energy conservation if there is no change in the spectrum of the light. And that means delta function in, delta function out.

Now I know I say it as if it would be the simplest thing in the world, but maybe you have questions. Does anybody want to maybe argue or defend why the light should be broadened? And actually discuss very soon other examples. We go to higher intensity, and at higher intensity other things end and other things are possible. But at the low intensity limit, do you have any suggestion what could cause the broadening of light?

AUDIENCE: I guess this is more if [INAUDIBLE] but then look at it as pinned down, but you're going to see a Doppler shift from the recoil as the light comes out. So it accepts the light, and then it moves something, so then it has some velocity. And then when it emits, it's going to be emitting in a random direction. And so its final velocity is going to be in some basically arbitrary direction, because the two [INAUDIBLE] boosts that you get from accepting and emitting.

PROFESSOR: OK now first of all, I wanted for this problem to assume that the mass of the atom is infinite. And at that point, the recoil energy has 1 over the mass term. The recoil energy is 0 and we can neglect it. So the limit I'm talking about is sound.

But you're right. If you want to bring in the recoil energy, what would then happen is if an atom absorbs a photon and emits it pretty much in the same direction, there is no momentum exchange. Then it will be close to the laser frequency. But then you will find, depending on the angle now, there is a shift between 0 and 2 recoil energies.

If you would, however, measure the light at a specific angle, you would still find a delta function, but you would find it shifted. But this is really a complication which involves now the mechanical effect of light and the recoil of a photon. It doesn't change the result. Energy conservation requires the spectrum to be, so to speak, a delta function in total energy. And what we have just discussed is that, depending on the scattering angle, maybe there's a kinetic energy to consider.

So what else can broaden the light? Yes?

AUDIENCE: Finite pulse time?

PROFESSOR: Yes. So things are very different if I use a pulse laser. But here I said the incident light is monochromatic, just one frequency. And therefore you don't have any time resolution in the experiment. The moment you have time resolution, you would introduce into the whole discussion Fourier [INAUDIBLE] and such.

But frankly, I ask this question every time I teach the course. And every time the vast majority of the class gives the wrong answer. Let me ask you something else which is maybe a little bit simpler, and that's the following. You have a harmonic oscillator which has an eigen frequency omega 0. You drive the harmonic oscillator at frequency omega L. What is the oscillation of the oscillator? Is it omega 0, the harmonic oscillator frequency? Or omega L, the drive frequency?

AUDIENCE: [INAUDIBLE].

PROFESSOR: Of course, the drive frequency. There may be a short transient, but if we do not switch on the drive suddenly, we avoid the transient. The steady state solution is the harmonic oscillator always oscillates at the drive frequency. Now, isn't the atom a harmonic oscillator?

In 8421 we discuss the oscillator strings. We discuss that the polarizability of the atom, the dipole moment, the AC stock shift can really be regarded as an oscillating dipole moment. So the atom for that purpose is nothing else than a driven harmonic oscillator. And when you drive the atom at a laser frequency, the dipole moment oscillates at which frequency? The laser frequency. Not the resonant frequency. And therefore, the light which is scattered-- think about it semi classically. This is like an antenna. It's an electron. It's a charge. It's a dipole moment which oscillates at the drive frequency. And the light, which is emitted by this accelerated charge is at the drive frequency.

The atom is, for that purpose, nothing else than a driven harmonic oscillator. And I'm saying that probably several times in that course, when you have a semi

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classical picture and a fully quantum picture, and they do not fully agree. I've never gone wrong with the semi classical picture. But myself and many of my students have come to very wrong answers by somehow holding on just to the quantum picture, and now seeing what the semi classical limit is.

You may think too much here in the atom not as a harmonic oscillator, but a two level system. The two level system gets excited, and then after it is excited, it emits at the natural frequency. But you've taken something to seriously here. It's a two level system, but you cannot read the system that you first populate a level, and then you emit, because the energy of the laser is not exactly the energy needed to populate the level.

So therefore you shouldn't do first order perturbation theory where you populate a level and then you wait and wait for the emitted photon. This is light scattering. It's a second order problem. One photon in, one photon out. And you have to conserve energy. Sorry for hammering on it, but it's really an example where you can maybe realize certain misconceptions, and I really want to make sure that you fully understand what's going on in this situation. Questions?

Well, then let's make it more interesting. Let's introduce higher intensity. So let me first maybe show you-- we've talked a lot about diagrams-- what the diagram for this process is. So we have the laser frequency, and then we have the photon emitted. And of course, the length of this vector, this frequency is the same as the laser frequency.

So let's move on now and talk about higher intensity. The high intensity case we discuss in great detail in 8421, but you pretty much I think know also from basic quantum physics what happens if you take a two level system and drive it strongly. If you drive it strongly, stronger than spontaneous emission, that means there is a limit where you can just forget about spontaneous emission in leading order, and the system is nothing else than a two level system coupled by a strong monochromatic drive.

So I assume you've all heard about this solution? Get the Rabi oscillation between

ground and excited state. So the high intensity limit are Rabi oscillations at the Rabi frequency. So what happens now to the spectrum of the emitted light if you are in the limit that you have Rabi oscillation of the atomic system?

And I want you to think classically or semi classically.

AUDIENCE: Three peaks.

PROFESSOR: Three peaks you generate side bends. Classically, you have an emitter, an oscillating charge. But now you want to throw in that the atom goes from the ground to the excited state, from the excited to the ground state. So, so to speak, when it's in the ground state, it cannot emit. When it's in the excited state, it can emit and such. So you should think about an oscillating dipole which is now intensity modulated. And you know when you take a classic light source which is monochromatic, but put on top of it an intensity modulation at frequency omega Rabi, the solution of that are three peaks. The carrier, which is the laser frequency, plus two side bends at omega Rabi.

So therefore, what we now expect for the spectrum is we have the laser frequency, and this is sort of our result for low intensity. However, when we-- in the limit of strong drive, and you will see in a moment where strong drive clearly comes in. So if you have this modulation at the Rabi frequency, you obtain side bends at the Rabi frequency.

And the fact that there is a structure with three peaks is actually the famous Mollow triplet. I know when I was a graduate student there was big excitement because people measured for the first time the Mollow triplet. You need high resolution lasers and such, and people were just ready to measure that. Now it seems something which we just mention in basic courses because it's such a basic phenomenon.

OK, but let's work a little bit more on that, namely the Rabi frequency is the Rabi frequency of the drive, or you can say the coupling matrix element. But when we have a detuning-- and this is what you want to discuss now-- you have to add the

detuning in quadrature. So this is a frequency at which the atomic population oscillates, and it is just semi classically the modulation of the atomic population which creates the side bends.

Now if you take that to the limit of large detuning, or small drive, small Rabi frequency, this becomes delta. So that actually means if you go to our stick diagram here, the laser frequency is, of course, detuned by delta from the atomic frequency. But the Rabi frequency is delta, and this is the Rabi frequency.

Maybe some people who pressed A and B feel now vindicated, because now you have a component of the emission spectrum, which is at the atomic resonance frequency. And you have a second peak, which is omega 0 minus 2 delta or 2 omega Rabi.

Question. How would you give a simple answer if I would ask you but what about energy conservation? I was hammering so much on energy conservation. How can we conserve energy by have a laser photon and emitting one at a different frequency?

AUDIENCE: [INAUDIBLE].

PROFESSOR: It's compensated with that. And now let me ask you another question. If you would scatter n photons and you do an experiment, you would expect that, due to Poisson fluctuation and such-- we've talked about fluctuations of intensity and squeezed light and all that. But unless you do something special, you would expect if you excite an atom it emits n photons. You experiment. You measure plus minus square root and you measure Poisson in statistics.

But now let's expect you observe n plus photons on the upper side bend. And you measure and you observe n minus photons on the lower side bend. Would you expect that n plus and n minus both have now Poisson fluctuations, or would you expect something else?

In other words, let me be very specific, if you count n plus-- we can make it a clicker question. So you have a counting experiment, and you have n plus photons in the

upper side bend and n minus in the lower side bend. And if you simply look at the variance, which is the square of the standard deviation from n plus, you find that it is Poisson distributed.

So the question which I have now is what is the variance between n plus minus n minus. Is it simply the variance of n plus plus the variance of n minus? The second answer I want to give you is 0. And the third answer is something else. So this is A. This is B. And this is C.

Yes. Energy conservation. So there shouldn't be any fluctuation. The emission of the upper side bend and the lower side bend have to be correlated. And you would immediately verify that if I show you what is the diagram. We have ground and excited state. We excite with a laser which has detuning. But then we emit a photon on the lower side bend. The second laser photon can now resonantly reach the excited state.

And so you see we have absorbed two photons from the laser beam. And what we have emitted in this diagram is one lower and one upper side bend. This is what happens in second order-- well, second order in the laser beam, but fourth order in the diagram. We emit four photons. And each time the system does it, goes through that, it fulfills energy.

Or if I call the upper side bend click and the lower side bend clack. The atom only makes click clack click clack clack click clack clack click clack. It never does click click clack clack. It cannot do two clicks because it has to fulfill this diagram, and they always come in pair. The two photons always come correlated.

OK, so we understand now the spectrum. We understand when the resonant photons emerge. They emerge in the high intensity limit due to Rabi oscillations. The question is now what is the widths of those peaks? And I'm want to short you that too, because no you realize some things are easy, but other things are harder. In order to get that, you have to solve the optical Bloch equations, but I want to show you in the next half hour how we can at least get the salient features of this. Colin, you have a question?

AUDIENCE: The picture you just offered about the [INAUDIBLE]. That's assuming that we're not depleting the carrier [INAUDIBLE]. Because if we're depleting the carrier, wouldn't we expect to see sort of like a high order, almost like a Bessel function.

PROFESSOR: All right. OK, yes.

AUDIENCE: First order on to the second [INAUDIBLE].

PROFESSOR: Yes, thanks for bringing that to our attention. There is an assumption. When I said we have a laser beam, I assume we have a laser beam which delivers zillions of photons in such a way that depletion doesn't play a role. What that means technically we replace a laser beam by a c number, and a c number never changes, therefore cannot be depleted.

But if you would use a very weak laser beam, we have to modify the answer. The extreme case would be if we use single photon sources, then of course we can never scatter two photons because there's only one photon in the source. But already if you would have hundreds or photons in your cavity and you put in a single atom, the scheduling of single photons would not cause major depletion effects. [INAUDIBLE]?

AUDIENCE: In regards to the semi classical [? fabrication ?], we said we have a Rabi frequency, and when it's excited it's likely to emit, and when it's in the ground state it's not likely to emit. So whatever frequency you emit should be modulated by your Rabi frequency, but aren't you most likely to emit when you have a dipole moment? Which would be in the intermediate states between e and g, in which case those occur at twice the Rabi frequencies I think.

PROFESSOR: OK, now you go to subtleties of a semi classical picture. Number one is-- I just want to say we're a little bit dangerous, on slippery slope here. One is I really discuss the spectrum here. When I discuss the spectrum, I don't know when the photon is emitted. So when I said we have a Rabi oscillation, I told you that the system is modulated, and this gives rise to side bends. But I can only spectrally solve the side bends if I fundamentally do not have the time resolution to measure when in the

Rabi cycle is the photon emitted. The moment I would localize when the photon is emitted, whether the atom is in the ground or in the excited state, if I have a time resolution better than the Rabi oscillation, I do not have the spectral resolution to resolve the side bends. So therefore the question when the photon is emitted is not compatible with observing the spectrum.

The second question, I think, can be addressed. And that's the following. I will show you in the next 45 minutes during this class that the steady states solution of the optical Bloch equations give us a rate of photon scattering which is simply gamma times the excited state fraction. So I think from that I would say the more you have atoms in the excited state, the larger is the scattering rate.

So therefore, the semi classical picture that you have only a dipole moment when you are halfway between ground and excited state is overly simplistic here. Also, I just want to give you one word of warning. It's really an important comment. And this is the following. The oscillating dipole moment is a picture which uses sort of the analogy with the atom and the harmonic oscillator. And I mean yeah, I really sort of was a bit provocative a few minutes ago when I said just regard the atom as a harmonic oscillator.

But you have to be careful. There is one fundamental limit between-- one fundamental difference between the harmonic oscillator and a two level atom. And the fact is the following. This is a two level atom, and this is a harmonic oscillator.

So the difference is the following. An harmonic oscillator can be excited. You can pump more and more energy in it. You can go to larger amplitude coherent states, and there is no non linearity.

The atom saturates. And this is the difference. Saturation. When saturation comes, when you put more than a few percent, let's say 50%, into the excited state, you have to be very, very, very careful with analogies with the harmonic oscillator.

So if you have most of the population in the ground state, a little bit here, and almost nothing here, then we can describe the atom as an harmonic oscillator. And

everything you glean from the model of an electron tethered with a spring is not only qualitatively, it's quantitatively correct if you throw in the oscillator strength of the atom.

But the moment you pile up more population here, and you just ask about 50-50, at least I would say immediately be careful. Do not take the harmonic oscillator fully seriously at this point, because there's a fundamental difference. In the harmonic oscillator, when you're 50% here, you would have things over there. The harmonic oscillator stays linear. But the two level system starts to saturate. Good.

So the question is now we understand the stick diagram of the Mollow triplet that we have always a line at the carrier. Let me just put marks here. We have the carrier and the resonance frequency, but the question is now what is the width of the spectrum. The general answer is rather difficult, but let me give it to you. And I will derive it from the optical Bloch equation. Let me discuss that one limit where the detuning is larger than anything else. And the other case is the resonant case.

In general we started out by saying we have a delta function at low intensity. If you now go to higher and higher intensity, you still sort of have the delta function from the lowest order diagram, but the higher order diagram become more important. And the spectrum is in general a superposition of three broadened peaks and the delta function. At low intensity, you only have the delta function. At high intensity, the weight of the elastic scatter the delta function goes to 0.

And now let's discuss the widths. And that shows you that even light scattering by a two level system, those very, very simple Jaynes Cummings model has interesting and maybe non intuitive aspect. The side bends have a width of gamma. The carrier has a width of 2 gamma. But when you are on resonance, the situation changes. The carrier has now a width of gamma and side bends have a width of 3/2 gamma. But if you look at it, the sum of the three widths is always 4 gamma. And you will understand that in a few minutes.

OK, so this was more qualitative discussion. Let's now work more quantitatively and describe the Mollow triplet. The Hamiltonian for our system is the atomic

Hamiltonian times the the z Pauli matrix. The z Pauli matrix is excited excited minus ground ground state. And because of the conservation of probability, it's 2 times the excited.

Also just mathematically it's just that. We have the three Hamiltonian of the radiation field, which is a dagger a. And then we have the coupling of a two level system, which is the product of the atomic dipole moment written as sigma plus plus sigma minus raising and lowering operator. Just to remind us, sigma plus is e g. And sigma minus is g e.

And then we have to multiply with a plus a dagger. If you want, the atomic part is the dipole moment, and this is the electric field both written as operators. So this is the electric field.

And if you look at the coupling term-- we've discussed it before. You have four terms. But in the rotating wave approximation, which we want to use here, we keep only the two resonant terms, which is when we raise an excitation in the atom, we lose an excitation in the light field. Or if you go from the excited to the ground state, we emit a photon.

So we have two terms which form the rotating wave approximation. The other two terms are highly of resonant and can often be neglected. So this rotating wave approximation becomes of course the better, the more we go in resonant, because in one term is fully resonant and the other one is very off resonant.

And in this case, we can simplify the Hamiltonian because the number of excitations is conserved. The number of excitations is the number of excitations in the photon field plus the number of excited atoms. So this is the operator which measures the number of excitations. If in a [INAUDIBLE] counter rotating term, this operator is conserved.

So if I introduce as usual the detuning-- this is what my notes say. It seems to have the opposite sign. I'm not sure if I made a sign error here or later, or if I changed the definition. Let's just move on and see how it works out. So if you now do the rotating

wave approximation, our Hamiltonian has now-- by introducing the operator N, the number of [INAUDIBLE] times h bar omega.

However, we have multiplied the N operator with the photon energy. If the energy is in the atom, we have made a mistake, which is delta, and we correct that by using the z Pauli matrix. And then our interaction term is a dagger times sigma minus plus a times sigma plus.

Let's now discuss what are the eigen states of this Hamiltonian In the simple case when the detuning is 0. And we have sort of two levels of eigen state. One is the ground state with n photons. And one is the excited state with n photons. 0, 1, 2. So we can label this by the photon number. And the excited state of course starts with a higher energy for 0 photon. This is the energy with one photon.

And in the case of 0 detuning, the two letters are degenerate. And if we introduce the interaction, if g is now non 0, the levels split. And the splitting is given by g. There's a factor of two.

But now if you have photons in the field, a dagger acting on the photon field gives n plus one. So therefore for one photon we have square root 2. In general, we get an expression which is square root n plus 1.

Or to write it mathematically, if we have strong mixing between an excited state with n atoms-- with n photons. And a ground state with n plus 1 photons, for the case of delta equals 0, it's just plus minus. You metric as a metric contribution times 1 over square root 2. And these are now with the states we label plus minus. The upper and lower state of each manifold. And n is the photon number in the excited state. n plus 1 is the photon number in the ground state.

And the energy of those states is the number of [INAUDIBLE] n plus 1. But then we have the resonant splitting g. And because of the matrix element of a and a dagger, we get square root n plus 1.

Now if you use very strong drive-- and this addresses Colin's question about depletion-- we use a laser beam, which is described by a Rabi frequency. This is

proportional to the electric field. So be introduce the Rabi frequency, which is given by that. And the laser beam has so many photons that we don't care about n plus 1 or scattering a few photons, so we simply replace n n plus 1 n minus 1, whatever contains n, by the z number, which is the Rabi frequency of the laser beam.

So this eventually means that we have too many folds. Plus, minus. Plus, minus. So we have two states plus minus, and they are periodic when we add one photon to the field. Periodic in n. The splitting is the Rabi frequency defined above. And now we realize that when we have spontaneous emission between the manifold for n minus 1 and the manifold n, we have two possibilities. To emit light on resonance. And we have then one possibility to emit an upper side bend. And one possibility to emit the lower side bend.

So what is called the dressed atom picture is nothing else than the solution of an atom driven by a strong monochromatic field. The atom is dressed with photons, and the eigen states are no longer just atomic eigen states. These are dressed eigenstates of the combined atomic system plus laser field. So this dressed atom picture explains the Mollow triplet.

And you can immediately generalize it if you want to arbitrarily tuning when the superposition states plus minus do not have equal weight. But it's what you've seen a million times, the diagonalization of a two by two matrix. Colin?

AUDIENCE: How does this explain the [INAUDIBLE]?

PROFESSOR: It does not. This is what I just wanted to say. It explains the Mollow triplet, but not the line widths. The line widths cannot be obtained perturbatively. So for that we have to discuss now the Bloch vector, which is a solution of the optical Bloch equation. So now I'm going to explain you not all glorious details, but the salient feature of the line bits. So the density matrix for two level atom can be written as-- let me back up a moment. The density matrix has four matrix elements. But if the sum of the diagonal matrix element is one conservation of probability, we have actually three independent matrix elements.

And those three independent matrix elements can be parametrized by what is called the Bloch vector. A lot of you have seen it. We also discuss it in great detail in 8421. But I'm giving you the definition. And sigma is the vector of Pauli matrices.

To be specific, the three components of the Bloch vector are as follows. The z comportment measures, you can say, the population inversion. The difference between ground and excited state population. And the x and y component measure the coherencies. Either the sum of the coherencies or the difference of the coherencies. And here is minus imaginary unit.

If you replace the density matrix or the matrix elements sigma ee, or ee, or gg by definition the optical Bloch equation turns into differential equation for r. It's now a differential equation for r. And let me just write it down. Let me just write down the Hamiltonian part.

This is the equation of motion for the density matrix. I will add on in a minute the relaxation part which comes from the master equation. This is just the Hamiltonian part. And this would result into an equation of motion for the Bloch vector, which has delta here minus delta here minus g. g and the rest of the matrix elements is 0. The Hamiltonian, using rotation matrices, can be written by a z rotation matrix plus g times the x rotation.

And if you actually look at the solution-- it's not a finite. It's an infinitesimal rotation. This matrix tells you that the detuning does a z rotation, and the drive is responsible for an x rotation.

Let me tell you what happens. You have seen the Bloch vector many cases. If you don't drive the system, the Bloch vector in the ground state is down. In the upper state it's up. If it's in between-- that's where Timo's dipole moment comes in-- we have a superposition of ground and excited state. And it rotates at the atomic frequency omega 0.

But we are in the rotating frame of the laser at omega, so in the rotating frame, when the frame rotates with omega. And omega 0 rotation becomes the rotation

with delta. So therefore, the free evolution of the atom is based on this Hamiltonian that the Bloch vector rotates with delta in the rotating frame of the laser. And it's a rotation around the z-axis.

If you drive the system now, we take the Bloch vector from ground to excited, there are Rabi oscillations. And this is now a rotation around the x-axis. Well, whether it's x or y is a convention of notation, and I've defined it such that the driven system is an x rotation. The free evolution is a z rotation. And that's all you have to know. This is the most general solution of two level system without dissipation that we have two rotation angles. One is the free rotation, which is the detuning. It is the z-axis. And the driven system rotates around the x-axis.

But now this is well known. This is boring. But now we want to add what the master equation gives us. And the master equation uses as its [? limb ?] platform where we have the sigma plus jump operator. And I told you that we have to use the following form. We'll hear more about it, probably not today, but on Friday. You know that this gives simply the optical-- this gives us simply the optical Bloch equations.

We discussed it on Monday that we have now a damping gamma term for the population. An excited state population decays with a rate of gamma. And the coherences, the off diagonal matrix elements, decay with a rate of gamma over 2, and we spent a long time discussing this vector of 2. So this would mean now that the equation for the Bloch vector has gamma 2 terms, which comes from the dampening of the coherences. And it has gamma, which is the damping of the population.

Well if that would be all, everything is damped to 0, but everything is in the end leads to the ground state population. So we have to add this term. It's just a different way of the optical Bloch equations.

We have discussed in great lengths this form of the optical Bloch equations. And the sigma plus sigma minus is because these are just simple matrices with one matrix element. You've seen the optical Bloch equation, I think. We did also that in 8421 21 that you have the derivative of the density matrix includes now damping of the off

diagonal matrix elements by gamma over 2 damping by the diagonal matrix elements by gamma. If you simply substitute the r vector for the matrix element, you get this equation in one step. These are now the optical Bloch equations written as a differential equation for the Bloch vector.

OK, sorry if that was confusing, but it's simple definition substitution no brainer. These are the optical Bloch equations written in terms of the Bloch vector and no longer in terms of the density matrix. And questions about that?

OK, because now I want to draw conclusions. Our goal is to understand the spectrum and the line widths of the Mollow triplet of the emitted light. So I have to make the connection. How do I make the connection from the Bloch vector to emitted light? Well, it's done by the dipole moment, because it is the oscillating dipole which is responsible for emitting and scattering light.

The dipole moment can of course be obtained from the solution of the optical Bloch equation for the density matrix. It is the trace of the density matrix times the operator we are interested in. And this involves the matrix element dge. The dipole moment has only matrix elements between excited and ground state. And ground and excited state. Trace rho other parentheses.

So this is the matrix element. Let me now write down the density matrix in its matrix element rho ge plus rho eg. But now we want to go from the rotating frame back to the lab frame. And the rotating frame rotates at omega. So now I have to put back e to the i omega t and e to the minus i omega t. This is sort of going back from the rotating frame to the lab frame.

And the picture of we want to use is-- the intuitive picture is to use the Bloch vector. So I'm expressing now those matrix elements by the component x and y of the Bloch vector. And since the x and y component was the sum or difference of diagram matrix element of the density the coherences. I get now cosine omega t and sine omega t.

So that tells us that one component is in phase-- well in phase means with respect

to the driving field we assume that the system is driven by classical field e0 cosine omega t. And now we find that the oscillating dipole moment, it's a driven harmonic oscillator. And there is a part which is driven in phase with the drive field, and one which is in quadrature.

So we have those two components. And the fact is now the following. If you have an oscillating dipole, this gives rise to immediate-- or more accurately, I should say scattered light. Remember, if you assume there is photon first emitted and then a resonant photon absorbed, this picture is wrong. You should rather think of scattering light, but the light scattering is induced by the oscillating dipole.

And what happens now is-- and this is why the Bloch vector and the Bloch sphere is such a wonderful picture. The dipole moment are the transverse component of the Bloch vector. So therefore, you can say that if you're interested in the spectrum of the emitted light, what you should find out is what is the spectrum of r.

Let me back up one step. If you want to get the spectrum of the emitted light from first principles-- I'm cheating a little bit here. I'm saying let's get it from the dipole moment. You should actually get it from the correlation function of the dipole moment. It's sort of the two time correlation function. But you can show-- and this takes another 10, 20, 30 pages in API.

The temporal correlation function of the dipole moment fulfils the same differential equation as the Bloch vector. So I've given you here sort of the intuitive link that would oscillates is the Bloch vector it emits it. But technically, because you may not control the phase of each atom, you should rather say the spectrum is a correlation function of the dipole, and we should look at the spectrum of the correlation function. But instead, I'm looking now at the spectrum of the optical Bloch vector. And you can show mathematically exactly that they obey the same differential equation, but it's a little bit tedious to do that.

OK, so therefore if you take this little inaccuracy-- forgive me this inaccuracy. Once we know what the optical Bloch vector is doing spectrally, we know the spectrum of the emitted light. And for that, for the optical Bloch vector, we simply look at this

matrix, and we ask what are the eigenvalues of this matrix? This matrix has three by three matrices, three eigenvalues. The real part of those eigenvalues gives us the position of the Mollow triplet. And the imaginary part gives us the widths of those peaks.

So if you look at this matrix-- and I'm not doing the complicated cases here. If you look at this matrix for delta equals 0 and g equals 0. Well it is already in diagonal form, so it has three imaginary eigenvalues minus gamma over 2 minus gamma 2 minus gamma.

So this matrix has three complex eigenvalues. And I want to discuss two cases. For this case we have minus gamma over 2 for the x and y component of the optical Bloch vector. And the z component is minus gamma. And this is the situation. Apart from a factor of two that we have gamma, 2 gamma, and gamma.

The fact of 2 comes of course. Are you asking what is the spectrum of the electric field or what is the spectrum of the power? If you go from e to e squared, then if you have an exponential decay with gamma of e, e squared decays with two gamma. So that's where factors of 2 come from.

AUDIENCE: Can you label the top of the diagram with [INAUDIBLE] just the delta equals 0.

PROFESSOR: Yes. Give me a second.

AUDIENCE: [INAUDIBLE] g equals 0, right?

PROFESSOR: The fact is here we're looking at limiting cases. And the more important thing is that this is the case of weak drive. So what is more important is that the drive frequency is 0. I'm not giving you the full picture here. The full glorious derivation is in API. And just tried to sort of entertain you by giving you a few appetizers. If you want, read it up there.

But what I want to show you is they're two simple cases, but it looks already intriguing because here this is wider than the side bends, and here it's the opposite. And I want to give you a taste of it. And I'm not doing it rigorously, and you're right, I

have to think about it. But the important part is that the drive frequency is 0. We are not driving, we are not rotating in x. The system is mainly rotating around the z-axis.

And at that moment, the matrix for the differential equation for the optical Bloch vector has three imaginary parts. It's gamma, 2 gamma, and gamma or half of it if you look at the amplitude. And now I want to just show you in a second how we can get the second part. But yeah, I'm guilty as charged. There are small gaps in my argument. But all I want to show you is how you find-- I want to just sort of convince you that the optical Bloch equations with this matrix have all the ingredients to explain it.

OK, so what I've done here is I've shown you that we have those three ingredients. And actually the moment-- I should have said it before answering your question. When we now crank up delta, what we're physically doing is we're doing a rapid rotation of the Bloch vector around the z-axis.

So when we rapidly around the z-axis, we're not doing anything to the z component, but we're strongly mixing the x and y component, but since [? half ?] the imaginary part-- I'm waving my hands now. We're not changing the imaginary part for x and y. And we still have two imaginary parts, which are gamma over 2. One imaginary part which is gamma.

I know I'm running out of time, but I only need a few more minutes. So if you do a rapid rotation around the z-axis, we obtain three eigenvalues. Since we rotate around the z-axis, we have minus gamma. Sorry, my notes are better than what I remember. I looked through the notes.

But what I wanted to say was the following. I first show you what are the eigenvalues of this matrix in one case, which is just a warm up. I'm telling you there are those three values. And now I'm saying we are interested in the physical situation of rapid rotation. The rapid rotation modifies the three eigenvalues. The three eigenvalues are no longer real. The rapid rotation around the z-axis adds an imaginary i delta to the x and y-axis.

Because now we are rotating, x and y are getting mixed. And this means now that we have three peaks. They are located in the rotating frame at 0. This is the carrier at plus minus delta. These are the two side bends. And this part are the widths of it. This is how you have to interpret the result. So we have to know three peaks. And the full widths of half maximum for the intensity, which is the amplitude squared is gamma, 2 gamma, and gamma.

And we can now immediately proceed to the next case. If we are on resonance, and we drive The system strongly, now we are driving the system strongly around, not the z-axis, around the x-axis. So now if we add the strong drive around the x-axis, we have now eigenvalues which is plus minus ig. This is the rotation we add, and this gives e to the i-- in rotation ge to the i omega t gives an ig.

And what happens is the following. The rapid rotation around the x-axis-- well, if you rotate around the x-axis, you're not touching x. And without any rotation, we had an eigenvalue for x which was minus gamma over 2. And if you take the matrix of whatever the system and rotate it around the x-axis, this is preserved.

AUDIENCE: Excuse me.

PROFESSOR: Yep, I need one more minute. Sorry. But the rotation around x strongly mixes y and z. So therefore the two other eigenvalues appear at rotation ig. And it is now the average of gamma and gamma over 2, which is 3/4 gamma. And if you go to the amplitude squared, it is 3/2.

So therefore we have the situation that we have those three eigenvalues. And this is responsible for the three peaks at plus minus g and 0. And the widths of it is I have to multiply by a factor of 2 gamma and 3/2 gamma. And this explains the two limiting cases I presented to you earlier. No time for question, the other people are waiting. Reminder we have class on Friday.