Lecture 5: Symmetries and Invariance Principles – Part I

Overview:

1. The Noether Theorem
2. Formal aspects of group theory
3. Lie groups and Lie algebras
4. SU(2)
5. Isospin

---

1. SU(3)
2. Discrete symmetries: P, C, G
3. CP – K-system
4. CPT

1. The Noether Theorem:

The Noether theorem plays a central role in theoretical physics. It allows to relate basic ideas of physics:

a) invariance of the form that a physical law takes with respect to any (generalized) transformation and

b) conservation law of a physical quantity.

Noether's theorem (Emmy Noether, 1917):

> To every symmetry, there is a corresponding conservation law and vice versa!

Examples:

1. Invariance of a physical system under translation:
   $\longrightarrow$ Conservation of momentum
2. Invariance of a physical system under rotation:
   $\longrightarrow$ Conservation of angular momentum
3. Invariance of a physical system under time:
   $\longrightarrow$ Conservation of energy

· The Noether theorem:

Define a set of transformations:

time : $t' = t'(t)$

space : $q_i' = q_i'(q_1, \dots, q_f, t)$ (1)

For the inverse operation:

$$t = t(t') \, ; \quad q_i = q_i(q_1', \dots, q_f', t') \tag{2}$$

A system is described by a given Lagrange function $L(q_1, \dots, \dot{q}_1, \dots, t)$. With the above transformations in (2), we get:

$$L(q_1, \dots, \dot{q}_1, \dots, t)\, dt = \underbrace{L'(q_1', \dots, \dot{q}_1', \dots, t')}_{\text{new Lagrangian } L' \text{ depending on } q_1', \dots, t'}\, dt' \tag{3}$$

Goal: Find the condition in which the equations of motion have the same form as in the old variables.

Such a transformation exists if the new Lagrange function $L'(q_1', \dots, \dot{q}_1', \dots, t')$ equals to the old Lagrange function $L(q_1', \dots, \dot{q}_1', \dots)$ or differs by the total differential of a function $\Omega(q_1', \dots, \dot{q}_1', \dots, t')$.

$$L'(q_1',\dots,\dot q_1',\dots,t') = L(q_1',\dots,\dot q_1',\dots,t') + \frac{d\Omega}{dt'}(q_1',\dots,t') \tag{4}$$

Together with equation (3) we find:

$$\boxed{L(q_1,\dots,\dot q_1,\dots,t)\,dt = L(q_1',\dots,\dot q_1',\dots,t')\,dt' + d\Omega(q_1',\dots,t')} \tag{5}$$

Let us define a set of transformations. Provided that the symmetry transformations satisfies a <u>continuous group</u>, it is sufficient to consider only <u>infinitesimal transformations</u> (we will come back to this later):

$$\boxed{\begin{aligned} t' &= t + \delta t \\ q_i' &= q_i + \delta q_i \\ \dot q_i' &= \dot q_i + \delta \dot q_i \end{aligned}} \tag{6}$$

Before we consider a transformation of that type on equation (5), let us derive a few important relations.

a) $\delta \dot{q}_i = \frac{d}{dt}\delta q_i - \dot{q}_i \frac{d}{dt}\delta t$:

$$\dot{q}_i' = \frac{d}{dt'} q_i' = \frac{d}{dt} q_i' \frac{dt}{dt'} = \frac{d}{dt}(q_i + \delta q_i)\frac{dt}{dt'}$$

now use:

$$\frac{dt}{dt'} = \frac{1}{\frac{dt'}{dt}} = \frac{1}{\frac{dt}{dt} + \frac{d}{dt}\delta t} = \frac{1}{1 + \frac{d}{dt}\delta t}$$

with: $\frac{1}{(1+x)} \simeq 1 - x$ for $x \ll 1$ we find

$$\delta \dot{q}_i' = \dot{q}_i' - \dot{q}_i = \frac{d}{dt}(q_i + \delta q_i)\frac{dt}{dt'} - \dot{q}_i =$$

$$= \left(\dot{q}_i + \frac{d}{dt}\delta q_i\right)\left(1 - \frac{d\delta t}{dt}\right) - \dot{q}_i =$$

$$\delta \dot{q}_i' = \dot{q}_i - \dot{q}_i \frac{d\delta t}{dt} + \frac{d\delta q_i}{dt} - \underbrace{\frac{d\delta q_i}{dt} \cdot \frac{d\delta t}{dt}}_{O(\delta^2)} - \dot{q}_i =$$

$$= \frac{d}{dt}\delta q_i - \dot{q}_i \frac{d\delta t}{dt}$$

therefore:

$$\boxed{\delta \dot{q}_i = \frac{d}{dt}\delta q_i - \dot{q}_i \frac{d}{dt}\delta t} \quad (7)$$

b.) $\delta(\mathcal{L}\,dt)$:

$$\delta(\mathcal{L}\,dt) = \delta\mathcal{L}\,dt + \mathcal{L}\,dt =$$

$$= \sum_i \left(\frac{\partial \mathcal{L}}{\partial q_i}\,\delta q_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \cdot \delta \dot{q}_i \right) dt + \left(\frac{\partial \mathcal{L}}{\partial t}\right) \delta t\,dt + \mathcal{L}\,d\delta t$$

With $\delta \dot{q}_i = \frac{d}{dt}\,\delta q_i - \dot{q}_i \frac{d}{dt}\,\delta t$ we get:

$$\boxed{\delta(\mathcal{L}\,dt) = \sum_i \left(\frac{\partial \mathcal{L}}{\partial q_i} \cdot \delta q_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \cdot \frac{d}{dt}\,\delta q_i \right) dt + \left(\frac{\partial \mathcal{L}}{\partial t}\right) \delta t\,dt + \left(\mathcal{L} - \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \cdot \dot{q}_i \right) \frac{d\delta t}{dt}\,dt} \tag{8}$$

With these two relations (equation 7 and 8) we can now consider the infinitesimal transformations in equation (5):

$$L(q_1, \dots, \dot{q}_1, \dots, t)\,dt = L(\underline{q_1 + \delta q_1}, \dots, \underline{\dot{q}_1 + \delta \dot{q}_1}, \dots, \underline{t + \delta t})\,d(\underline{t + \delta t}) + d\,\delta\Omega(q_1, \dots, t)$$

We can rewrite this as:

$$L(q_1, \ldots, \dot{q}_1, \ldots, t)\, dt = L(q_1 + \delta q_1, \ldots, \dot{q}_1 + \delta \dot{q}_1, \ldots, t + \delta t)\, dt$$
$$+ L\, \delta t + d\, \delta \Omega$$

$$\delta L\, dt + L\, dt + d\, \delta\Omega = 0$$

Therefore:

$$\boxed{\delta(L\, dt) + d\,\delta\Omega = 0} \qquad (9)$$

- Plug now equation (8) into (9):

$$\boxed{\begin{aligned}&\sum_i \left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt}\delta q_i \right) + \left(\frac{\partial L}{\partial t}\right) \cdot \delta t \\ &\qquad + \left(L - \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right) \frac{d\delta t}{dt} = - \frac{d\,\delta\Omega}{dt}\end{aligned}} \qquad (10)$$

Use now the following equations to formulate an equation as: $\frac{d}{dt}[\ldots] = 0$ → const.

1. $$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$$

2. $$\frac{d}{dt}\left(L - \sum \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right) = \left(\frac{\partial L}{\partial t} \right)$$

° We now :

1.

$$\frac{d}{dt}\left\{\left(L-\sum_i \frac{\partial L}{\partial \dot q_i}\,\delta \dot q_i\right)\delta t\right\} =$$

$$\frac{d}{dt}\underbrace{\left(L-\sum_i \frac{\partial L}{\partial \dot q_i}\cdot \dot q_i\right)}_{\frac{\partial L}{\partial t}\cdot\delta t}\delta t + \left(L-\sum_i \frac{\partial L}{\partial \dot q_i}\cdot \dot q_i\right)\frac{d\,\delta t}{dt}$$

2. With: $\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right)$ we find :

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\,\delta q_i\right) = \frac{\partial L}{\partial q_i}\cdot\delta q_i + \frac{\partial L}{\partial \dot q_i}\cdot\frac{d}{dt}\cdot\delta q_i$$

this provides the following relation for $\frac{d}{dt}\left[\,..\,\right]=0$

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot q_i}\cdot\delta q_i + \left(L-\sum_i \frac{\partial L}{\partial \dot q_i}\cdot \dot q_i\right)\delta t + \delta\Omega\right] = \phi$$

That means:

$$\boxed{\sum_i \left(\frac{\partial L}{\partial \dot q_i}\right)\cdot\underline{\delta q_i} + \left(L-\sum_i \frac{\partial L}{\partial \dot q_i}\cdot \dot q_i\right)\underline{\delta t} + \underline{\delta\Omega} = \text{const.}}$$

Noether Theorem (11)

- Let's go back to our examples:

1. Translation: $\delta x_3 = \text{const.}$; $\delta x_1 = \delta x_2 = 0$; $\delta t = 0$

$\delta \Omega = 0$

this gives with equation (11):

$$\boxed{\frac{\partial L}{\partial \dot{x}_3} = p_3 = \text{const.}}$$

Conservation of momentum

2. Rotation: $\delta \varphi_3 = \text{const.}$; $\delta \varphi_1 = \delta \varphi_2 = 0$; $\delta t = 0$; $\delta \Omega = 0$

$$\boxed{\frac{\partial L}{\partial \dot{\varphi}_3} = l_3 = \text{const.}}$$

Conservation of angular momentum

3. Time: $\delta t = \text{const.}$; $\delta q_i = 0$; $\delta \Omega = 0$

$$\boxed{\sum_i \frac{\partial L}{\partial \dot{q}_i} \cdot \dot{q}_i - L = E = \text{const.}}$$

Conservation of energy

2. Formal aspects of group theory:

Group theory is the branch of mathematics that underlies the treatment of symmetry.

- Example: Rotation group

Given are a set of rotations R_1, R_2 and R_3. The set of rotations form a group. Each rotation is an element of the group.

Definition: A group is a set G on which a multiplication operation $\cdot$ is defined with the following properties:

1. If R_i and R_j are in G, $R_i \cdot R_j$ is in G
(closure)

2. There is an identity element I in G such that
$I \cdot R_i = R_i \cdot I = R_i$ for any R_i in G (Identity)

3. For every R_i in G, there is an inverse element in G called R_i^{-1} such that: $R_i \cdot R_i^{-1} = R_i^{-1} \cdot R_i = I$
(Inverse)

4. For every R_i, R_j, R_K in G:

$(R_i \cdot R_j) \cdot R_j = R_i \cdot (R_j \cdot R_K)$ (Associativity)

· **Note:** A group is called non-Abelian if the following holds: $R_i \cdot R_j \neq R_j \cdot R_i$ and vice versa.

1. Transformations in space and time form an Abelian group.
2. Rotations form a non-Abelian group

· **Groups can be:**

a) finite or infinite

b) continuous or discrete

The rotation group is a continuous group in that each rotation can be labeled by a set of continuously varying parameters $(\alpha_1, \alpha_2, \alpha_3)$.

The rotation group is a Lie group. The rotation can be expressed as the product of a succession of infinitesimal rotations $\longrightarrow$ The group is completely defined by the "neighborhood of the identity".

In quantum mechanics, a transformation of the system is associated with a unitary operator U in Hilbert space:

$$\boxed{|\psi\rangle \longrightarrow |\psi'\rangle = U|\psi\rangle}$$

• Review on Matrix algebra:

1. Unitary matrix:

A square matrix U is a Unitary matrix if $U^* = U^{-1}$ where U^* denotes the adjoint matrix and U^{-1} is the inverse matrix.

2. Adjoint matrix:

The adjoint matrix, sometimes called the adjugate matrix, Hermitian transpose, is defined by $U^* = \bar{U}^T$

where U^T denotes the transpose of matrix U (replace matrix elements U_{ij} by U_{ji}) and $\bar{U}$ denotes the conjugate matrix (replace matrix U_{ij} by the complex conjugate $\overline{U_{ij}}$).

3. Hermitian matrix:

A square matrix H is called Hermitian if it is self-adjoint:

$$H = H^*$$

Example: Pauli matrices

4. orthogonal matrix:

a $n \times n$ matrix O is an orthogonal matrix, if

$$A \cdot A^T = \mathbb{1} \quad , \text{ i.e. } \quad A^{-1} = A^T \quad \underline{\text{with:}} \quad (a^{-1})_{ij} = a_{ji}$$

Note:

1. A unitary matrix U is called special unitary matrix, if:

$$U U^* = \mathbb{1} \quad \text{and} \quad \det U = 1 \qquad SU$$

2. A orthogonal matrix O is called special orthogonal matrix, if:

$$O \cdot O^T = \mathbb{1} \quad \text{and} \quad \det O = 1 \qquad SO$$

Important groups in elementary particle physics:

Group name:	matrices in group:
$U(n)$	$n \times n$ unitary
→ $SU(n)$	$n \times n$ unitary w/ determ. $\underline{1}$
$O(n)$	$n \times n$ orthogonal
$SO(n)$	$n \times n$ orthogonal w/ determ. $\underline{1}$

A transformation group of a quantum mechanical system is associated with a mapping of the group into a set of unitary operators.

For each x in G, there is a $U(x)$ which is a unitary operator: $x \to U(x)$

group operations are preserved: $U(x) \cdot U(y) = U(x \cdot y)$

Such a mapping is called representation.

example: $U(n) = e^{in\theta}$ is a representation of the additive group of integers:

$$e^{in\theta} \cdot e^{im\theta} = e^{i(n+m)\theta}$$

Note: 1. It is convenient to view representations as abstract linear operators and as matrices.

2. Two representations U_1 and U_2 are equivalent if they are related by a similarity transformation: $U_2 = S U_1 S^{-1}$

3. A representation U is <u>reducible</u> if it is equivalent to a representation U' with <u>block-diagonal</u> form:

$$\boxed{U' = S U S^{-1} = \begin{vmatrix} U'_1(x) & 0 \\ 0 & U'_{2}(x) \end{vmatrix}}$$

4. The representation U' is said to be the direct sum of U'_1 and U'_2:

$$\boxed{U' = U'_1 \oplus U'_2}$$

5. A <u>representation</u> is <u>irreducible</u> if it is not reducible, that is if it cannot be put into block diagonal form by a <u>similarity transformation</u>.

→ We will almost never talk about the group elements as abstract mathematical objects, but in terms of their <u>representations</u>: operators ("matrix")

3. Lie-groups and Lie algebras:

Compact Lie groups are groups of unitary operators in which the group elements are labeled by a set of continuous parameters.

Any unitary matrix can be written as:

$$\boxed{U = e^{iH} = 1 + iH - \frac{1}{2!} \cdot H^2 + \dots}$$

H: Hermitian, traceless matrix

In a Lie group, the elements of the group are characterised by a finite number of real parameters a_α. For $SU(n)$ one has n^2-1 real parameters, the number of independent parameters for an arbitrary, traceless, Hermitian matrix.

$$\boxed{H = \sum_{\alpha=1}^{N} a_\alpha \cdot L_\alpha}$$

Parameters   generators

∘ Note:

Do not mix up dimension of L_α from dimension n^2-1!

e.g.: $SU(2)$

$N = n^2-1 = 3$

$\longrightarrow$ 3 generators!

- Again: For $SU(2)$: 3 generators
  The dimension of those generators depends on the quantum mechanical system under consideration:
  - spin $\frac{1}{2}$ particles
  - spin 1 particles

In general:

To study the representations, it is sufficient to study the generators:

$$\boxed{[L_\alpha, L_\beta] = i\, C_{\alpha\beta\gamma}\, L_\gamma}$$

The generators and their commutation relations specify a Lie algebra where the $C_{\alpha\beta\gamma}$ are called the structure constants.

Jacobi identity: $\boxed{[L_\alpha, [L_\beta, L_\gamma]] + \text{cycl. perm.} = 0}$

$\longrightarrow$ Simplest non-Abelian Lie algebra

$N = 3$, $SU(2)$: $\quad C_{\alpha\beta\gamma} = \varepsilon_{\alpha\beta\gamma}$

anti-sym. tensor

4. SU(2):

1. General:

We first outline the construction of representations of su(2) in a systematic way. We want to construct the Hermitian representation matrices $\vec{S} = (S_1, S_2, S_3)$ that satisfy:

$$\boxed{[S_i, S_j] = i\, \varepsilon_{ijk} S_k}$$

Casimir operator:

Except for operators from the set of generators there are other operators that can be constructed from them and commute with all generators of the group so called Casimir operators:

$$\boxed{S^2 = S_x^2 + S_y^2 + S_z^2}$$

- Explicit construction from the commuting rules:

$$\boxed{\begin{aligned} S_\pm &= S_x \pm i S_y \\ [S_z, S_\pm] &= \pm S_\pm \\ [S_+, S_-] &= 2 S_z \\ [S^2, S_i] &= 0 \end{aligned}}$$

$$S^2 | s, m \rangle = s(s+1) \, | s, m \rangle$$
$$S_z | s, m \rangle = m \, | s, m \rangle$$

spin s

Note: $m = -s, -s+1, \dots, s$
s can take on any value $0, \frac{1}{2}, 1, \dots$

Matrix representation:

a) singlet: 1-dimensional representation: spin 0

$$| 0, 0 \rangle$$

$$S_z = (0) \, ; \quad S_+ = (0) \, ; \quad S_- = (0)$$

b.) 2 dimensional repr. : spin $\frac{1}{2}$

For $SU(2)$, the 2-dim. repr. has the basis states

$$|s=\tfrac{1}{2}, m=\tfrac{1}{2}\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |s=\tfrac{1}{2}, m=-\tfrac{1}{2}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

"spin up" "spin down"

• representation matrices:

$$S_z = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \quad S_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad S_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

generators of $SU(2)$ for spin 1/2

c.) 3 dim. repr. : spin 1

$$|1,-1\rangle \quad |1,0\rangle, \quad |1,1\rangle$$

matrix representation:

$$S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad S_+ = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \quad S_- = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

generators of $SU(2)$ for spin 1

· combining representations:

$$\hat{S} = \hat{S}_A + \hat{S}_B \longrightarrow |s_A, m_A\rangle \; ; \; |s_B, m_B\rangle$$

$$S = |s_A - s_B|, |s_A - s_B| + 1, \ldots, s_A + s_B$$

$$M = m_A + m_B$$

$$\boxed{|s_A s_B, S M\rangle = \sum_{m_A, m_B} C(m_A m_B; S M) \, |s_A \, s_B \, m_A \, m_B\rangle}$$

Clebsch-Gordon coefficient

· other useful notation for combined representation:

a.) $2 \otimes 2 = 3 \oplus 1$ or $\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$

2 spin $\frac{1}{2}$ system : 2-dim-repr.

irred. repr.

b.) for 3 spin $\frac{1}{2}$ particles:

$$2 \otimes 2 \otimes 2 = (3 \otimes 2) \oplus (1 \otimes 2) = 4 \otimes 2 \otimes 2$$

$$\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = (1 \otimes \tfrac{1}{2}) \oplus (0 \otimes \tfrac{1}{2}) = \tfrac{3}{2} \otimes \tfrac{1}{2} \otimes \tfrac{1}{2}$$

5. Iso spin: $\longrightarrow$ SU(2)

Heisenberg observed shortly after the discovery of the neutron in 1932 that the neutron is almost equal to the proton apart from their respective charge.

$$m_p = 938.28\,MeV/c^2 \qquad m_n = 939.57\,MeV/c^2$$

Heisenberg proposed that one regards neutrons and protons as "two states" of a single particle, the nucleon:

$$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \; ; \quad n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

iso spin I with 3 generators: I_1, I_2, I_3

$$p = \left| \tfrac{1}{2}\ \tfrac{1}{2} \right\rangle \qquad n = \left| \tfrac{1}{2}\ -\tfrac{1}{2} \right\rangle$$

Now: Strong force is invariant under rotations in iso spin space

$\longrightarrow$ Noether theorem: Iso spin is conserved!

1.

nucleon: two dimensional repr. $SU(2)$ isospin $\frac{1}{2}$

$I = 1/2$

2.

Pions: $I = 1$ 3-dim repr.

$$\pi^+ = |1\,1\rangle \; ; \; |1\,0\rangle \; ; \; \pi^- = |1\,-1\rangle$$

3. Λ, $I = 0$: 1-dim. repr.

$$\Lambda = |0,\,0\rangle$$

4 Δ's, $I = 3/2$: 4-dim. repr.

$$\Delta^{++} = \left|\tfrac{3}{2}\,\tfrac{3}{2}\right\rangle \; ; \; \Delta^{+} = \left|\tfrac{3}{2}\,\tfrac{1}{2}\right\rangle \; ; \; \Delta^{0} = \left|\tfrac{3}{2}\,-\tfrac{1}{2}\right\rangle \quad \Delta^{-} = \left|\tfrac{3}{2}\,-\tfrac{3}{2}\right\rangle$$

general: multiplicity: $2I + 1$

generators: isospin operators

$$\hat{T}_i = \frac{1}{2}\tau_i \quad (i = 1,2,3) \qquad \tau_i : \text{Pauli Matrix}$$

$$\boxed{\left[\hat{T}_i, \hat{T}_j\right] = i\,\varepsilon_{ijk}\,\hat{T}_k}$$

Compound systems : isospin

2 - nucleon system

$|1\,1\rangle = |pp\rangle$

$|1\,0\rangle = (1/\sqrt{2})\,(pn + np)$

$|1\,{-1}\rangle = nn$

$|0\,0\rangle = (1/\sqrt{2})\,(pn - np)$ deuteron : isosinglet

Dynamical importance of isospin invariance:

deuteron : $I = 0$ nucleon - nucleon scattering

- $p + p \rightarrow d + \pi^+$
  $|1\,1\rangle \quad |1\,1\rangle$
- $p + n \rightarrow d + \pi^0$
  $|1\,0\rangle \quad (1/\sqrt{2})\,(|1\,0\rangle + |0\,0\rangle)$
- $n + n \rightarrow d + \pi^-$
  $|1\,{-1}\rangle \quad |1\,{-1}\rangle$

only $I = 1$ contribute

$M_a : M_b : M_c = 1 : (1/\sqrt{2}) : 1$

since: cross-section $\propto |M|^2$

$$\boxed{\longrightarrow \quad \sigma_a : \sigma_b : \sigma_c = 2 : 1 : 2}$$