Chapter 2: Deriving AdS/CFT

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Hong Liu, Fall 2014

Lecture 15

Important equations for this lecture from the previous ones:

1. The spacetime symmetry preserved by the existence of a Dp-branes (or N of them coincidentally), equation (3) in lecture 13:
$$\left(T^p \otimes SO^+(1,p)\right) \otimes SO(D-1-p) \tag{1}$$

2. The relation between the gravitational constant G_N and string theory's g_s and α', equation (15) in lecture 12:
$$G_N = 8\pi^6 g_s^2 \alpha'^2 \tag{2}$$

2.1.3: D-BRANES (cont.)

Before we end this section, let's review about D-branes in superstring theory.

D-branes in bosonic string always have open string tachyonic modes, therefore it's unstable. The decaying is partially understood from string field theory, as one can see tachyons as the decaying product of the D-branes. In superstring theory, however, D-branes of certain dimensions are stable (can be seen from the consistency of stringy interactions and also the conserved charge associated with the nonperturbative object) and do not have tachyons. For these D-branes:

1. They carry a conserved charge

2. The worldvolume theory for D-branes is supersymmetric, and in addition to A_α and Φ^a there are also massless fermions.

3. The bosonic part of massless closed super string spectrum has $\Phi, B_{\mu\nu}, h_{\mu\nu}$; for IIA superstring we also get $C_\mu^{(1)}$ and $C_{\mu\nu\rho}^{(3)}$, for IIB superstring we also need χ, $C_{\mu\nu}^{(2)}$ and the self-dual $C_{\mu\nu\lambda\rho}^{(4)+}$. The anti-symmetric potentials can be thought as the generalizations of the familiar Maxwell field A_μ. In exterior form calculus:
$$A = A_\mu dx^\mu \ ; \ F = dA \ , \ \mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \ ; \tag{3}$$
$$C^{(n)} = C^{(n)}_{\mu_1\ldots\mu_n} dx^{\mu_1}\ldots dx^{\mu_n} \ ; \ F^{(n+1)} = dC^{(n)} \ , \ \mathcal{L} = -\frac{1}{2n!}F^{\mu_1\ldots\mu_{n+1}}F_{\mu_1\ldots\mu_{n+1}} \tag{4}$$
The gauge symmetry transformation is $C^{(n)} \to C^{(n)} + d\Lambda^{(n-1)}$ for any (n-1)-form $\Lambda^{(n-1)}$. For the 1-form A_μ, the source is a point particle, as it couples with the object through the action with worldline trajectory C:
$$S_0 \sim \int_C A = \int_C A_\mu \frac{dX^\mu}{d\lambda} d\lambda \tag{5}$$

In the same fashion, source for a $C^{(p+1)}$ is a p-dimensional object (can be a Dp-brane) with worldvolume trajectory Σ:

$$S_p \sim \int_\Sigma C^{(p+1)} = \int_\Sigma C^{(p+1)}_{\mu_1 \cdots \mu_{p+1}} \frac{\partial X^{\mu_1}}{\partial \lambda_1} \cdots \frac{\partial X^{\mu_{p+1}}}{\partial \lambda_{p+1}} d^{p+1}\lambda \tag{6}$$

$X^\mu(\lambda_1 ... \lambda_{p+1})$ is the embedding of Σ (C^{p+1} pull-back to the worldvolume Σ of the Dp-brane) in spacetime, and $\lambda_{1,..,p+1}$ are the worldvolume coordinates of a Dp-branes. The gauge symmetry of $C^{(p+1)}$ gives rise to a conserved charge, and an object that carrying the minimal possible charge must be stable. One also can dualize a n-form potential:

$$d\tilde{C}^{(D-2-n)} = \star dC^{(n)} \tag{7}$$

Then a $(D-3-n)$-brane can couple to the field $\tilde{C}^{(D-2-n)}$, or, magnetically couple to the original field $C^{(n)}$.

Thus:

1. For type IIA superstring, $C^{(1)}$ couples electrically to D0-branes and magnetically to D6-branes, $C^{(3)}$ couples electrically to D2-branes and magnetically to D4-branes.
2. For type IIA superstring, χ couples electrically to D(-1)-branes and magnetically to D7-brance, $C^{(2)}$ couples electrically to D1-branes and magnetically to D5-branes, self-dual $C^{(4)+}$ (with $F^{(5)+} = \star F^{(5)+}$) couples both electrically and magnetically to D3-branes.

On these branes one gets the supersymmetric Yang-Mills (SYM) theories as a low energy effective description. We will be interested in the case of D3-branes (type IIB superstring), which results in the $\mathcal{N} = 4$ SYM theory in 4D.

2.2: D-BRANES AS SPACETIME GEOMETRY

D-branes gravitate and deform the spacetime geometry around them.

Consider a charged particle sitting at the origin of a 4D spacetime:

$$\mathcal{L} = \frac{1}{16\pi G_N} R - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \tag{8}$$

The equations of motion:

$$\partial_\mu F^{\mu\nu} = J^\nu \quad , \quad R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N \Big(T_{\mu\nu}(A) + T_{\mu\nu}(\text{particles}) \Big) \tag{9}$$

A point q charge particle of mass m inside spacetime gives:

$$J^0 = q\delta^{(3)}(\vec{x}) \quad , \quad T^{00} = m\delta^{(3)}(\vec{x}) \tag{10}$$

The gauge field configuration around the object:

$$A_0 = \frac{q}{4\pi r} \quad , \quad \int_{S^2} \star F = q \tag{11}$$

The spacetime around the object (choose a spherical symmetric coordinates) has the Reissner-Nordstrom metric:

$$ds^2 = -f(r)dt^2 + h(r)\Big(dr^2 + r^2 d\Omega^2\Big) \quad , \quad f(r) = \frac{1}{h(r)} = 1 - \frac{2m}{r} + \frac{q^2}{r^2} \tag{12}$$

The low effective theory of type IIB superstring theory is type IIB SUGRA. The regime of validity is when $g_s \ll 1$ (as the string loop-corrections is small), the energy scale $\ll \frac{1}{\alpha'}$ (the massive modes is effectively decoupled from the effective theory), and also the curvature of spacetime $\ll \frac{1}{\alpha'}$. Let's keep the energy or curvature scale of interests fixed, hence let $\alpha' \to 0$ (and $g_s \to 0$, obviously).

A D3-branes is both electrically and magnetically charged under the self-dual $C^{(4)+}$, and since $F^{(5)+} = \star F^{(5)+}$ therefore the generalized Dirac quantization of electric and magnetic charged $q\tilde{q} = 2\pi n$ ($n \in \mathbb{N} - \{0\}$) means a single D3-branes has electric charge and magnetic charge $q = \tilde{q} = \sqrt{2\pi}$. For N D3-branes, this gives the total charge to be $\sqrt{2\pi}N$.

From perturbative string interactions, it can be shown that the D-branes tension equal to the charge density up to a (known) prefactor, so that for total of N D3-branes, using equation (2):

$$T_3 = N\frac{q}{\sqrt{16\pi G_N}} = \frac{N}{(2\pi)^3 g_s \alpha'} \tag{13}$$

Consider a perturbative string scattering process with a D-brane, then using the doubling trick one can show that the boundary naturally given by D-brane fuses 2 conserved supercharges of the theory without the boundary become 1, hence the object only preserved half of the SUSY. This indicates that D-brane is a BPS state in the stringy Hilbert space, which gives a nontrivial relation between the D-brane tension and its charge.

From the spacetime symmetries preserved by a D3-brane, given in equation (1), one can guess the form of the spacetime metric deformed by a stack of N D3-branes (sitting at position r = 0):

$$ds^2 = f(r)\Big(- dt^2 + \sum_{i=1}^{3} dx_i^2\Big) + h(r)\Big(dr^2 + r^2 d\Omega_e^2\Big) \tag{14}$$

The SUGRA solution gives:

$$f(r) = \frac{1}{h(r)} = H^{-1/2}(r) \quad , \quad H(r) = 1 + \frac{R^4}{r}^4 \quad , \quad R^4 = N\frac{4}{\pi^2}G_N T_3 = N4\pi g_s \alpha' \tag{15}$$