8.821/8.871 Holographic duality

MIT OpenCourseWare Lecture Notes

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Lecture 2

1.2: BLACK HOLE THERMODYNAMICS

1.2.1: IMPORTANT SCALES

Planck scale

We can construct physical units using fundamental constants $\hbar$ (reduced Planck constant), G_N(gravitational constant), c (speed of light):

$$m_p = \sqrt{\frac{\hbar c}{G_N}} \approx 1.2 \times 10^{14} GeV/c^2 = 2.2 \times 10^{-5} g;$$
$$l_p = \sqrt{\frac{\hbar G_N}{c^3}} \approx 1.6 \times 10^{-33} cm;$$
$$t_p = \frac{l_p}{c} \approx 5.4 \times 10^{-44} s$$

This quantity is called "Planck scale", and represents the energy scale at which the quantum effects of gravity become strong.

Strength of gravity

Let us first compare the strength of gravity and strength of electro-magnetic (EM) interaction. In the EM case, interaction takes the form $V_{EM} = \frac{e^2}{r}$. We take the reduced Compton wavelength $r_c = \frac{\hbar}{mc}$ to be the smallest distance between particles, because this distance can be thought as the fundamental limitation on measuring the positions of a particle, taking quantum mechanics and special relativity into account. Using the unit of particle static mass, the EM interaction has the effective strength:

$$\lambda_{EM} = \frac{V_{EM}(r_C)}{mc^2} = \frac{e^2}{\hbar c} = \alpha = \frac{1}{137}$$

On the other hand, we can also get the effective strength of gravity:

$$\lambda_G = \frac{V_G(r_c)}{mc^2} = \frac{G_N m^2}{\hbar/mc}\frac{1}{mc^2} = \frac{m^2}{m_p^2} = \frac{l_p^2}{r_c^2}$$

Then $\lambda_G \ll 1$, for $m \ll m_p$. For example, in the case of electron, $m_e = 5 \times 10^{-4} GeV/c^2$, we have

$$\frac{\lambda_G}{\lambda_{EM}} \sim 10^{-43}$$

The gravity effect is quite weak in this case. But if the mass is at Planck mass scale m_p, then $\lambda_G \sim O(1)$, which means quantum gravity effects become significant (the corresponding length scale will be l_p).

Schwarzschild radius

Now taking a step back from the quantum gravity effects, we can even ask a simpler question: for an object of mass m, at what distance r_s from it, the classical gravity becomes strong? To answer this question, we can consider a probe mass m', then the classical gravity becomes strong means that

$$\frac{G_N m m'/r_s}{m'c^2} \sim 1 \quad \Rightarrow \quad r_s \sim \frac{G_N m}{c^2}$$

So now for an object of mass m, we have two important scales:

$$r_c = \frac{\hbar}{mc}: \quad \text{Reduced Compton wavelength}$$
$$r_s = \frac{2G_N m}{c^2}: \quad \text{Schwarzschild radius}$$

The pre-factor 2 of r_s comes from a GR computation of a Schwarzschild black hole.

From $\frac{r_s}{r_c} \sim \frac{m^2}{m_p^2}$, we can conclude

1. $m \gg m_p$, $r_s \gg r_c$: classical gravity (quantum effects not important);
2. $m \ll m_p$, $r_s \ll r_c$: r_s is not relevant, gravity effect is weak and not important;
3. $m \sim m_p$, $r_c \sim r_s$, quantum gravity effects are important.

If this were the whole story, life would be much simpler, but much less interesting. However, black holes can make quantum gravity effects manifest at macroscopic level, at length scales of $O(r_s)$, we will discuss this later.

Remark: l_p can be thought as the minimal localization strength. In non-gravitational physics, the probing length scale $l \sim \frac{\hbar}{p}$, in principle, can be as small as one wants if one is powerful enough to get sufficiently large p. But with gravity, when $E \sim p \gg m_p$, then $r_s \sim \frac{G_N p}{c^3}$ takes over as the minimal scale. Since $r_s \propto p$, so larger energies give larger length scales, l_p is the minimal scale one can probe. Alternatively, consider uncertainty principle $\delta p \sim \frac{\hbar}{\delta x}$, then $\delta x > \frac{G_N \delta p}{c^3} \sim \frac{G_N \hbar}{c^3}\frac{1}{\delta x}$, so we obtain $\delta x > \sqrt{\frac{\hbar G_N}{c^3}} = l_p$.

Now let us summarize various regimes of gravity for fixed energy scales we are interested in:

- Classical gravity: $\hbar \to 0$, G_N finite;
- QFT in a fixed spacetime (including curved): $\hbar$ finite, $G_N \to 0$;
- Quantum gravity: G_N, $\hbar$ finite; and in the semi-classical regime for quantum gravity, we take $\hbar$ finite and expand the theory in G_N.

1.2.2 Classical black hole geometry

Black hole geometry is the solution of Einstein equation with zero cosmology constant. The spacetime is due to an object of mass M. If we consider the object to be spherically symmetric, non-rotational, neutral, we have the Schwarzschild metric solution:

$$ds^2 = -f dt^2 + \frac{1}{f}dr^2 + r^2(d\theta^2 + \sin\theta^2 d\phi^2), \quad f = 1 - \frac{2G_N M}{r} = 1 - \frac{r_s}{r} \tag{1}$$

Note that from now on, we have adapted the convention to take $c = 1$.

The event horizon is defined at

$$r = r_s = 2G_N M$$

where $g_{tt} = 0$, $g_{rr} = \infty$. When r goes across the event horizon, f changes sign, r and t switch role.

Here are some simple facts on this metric:

1. It is time-reversal invariant, *i.e.* invariant under $t \to -t$.
   It does not describe a black hole formed from gravitational collapse which is clearly not time-reversal symmetric, but it is a mathematical idealization of such a black hole.
2. The spacetime is non-singluar at the horizon, as one can check this by computing curvature invariants. It is only a coordinate singularity (not an intrinsic singularity), where t (Schwarzschild time) and r coordinates become singular at the horizon.
3. At $r = r_s$, the surface is a null hypersurface.

4. The horizon is a surface of infinite redshift.
Consider an observer O_h at the hypersurface $r = r_h \approx r_s$ and another observer O_∞ at the hypersurface $r = \infty$. At $r = \infty$: $ds^2 \to -dt^2 + dr^2 + r^2 d\Omega_2^2$, t is the proper time for O_∞. On the other hand, at $r = r_h$: $ds^2 = -f(r_h)dt^2 + \cdots = -d\tau_h^2 + \cdots$. We have $d\tau_h = f^{1/2}(r_h)dt$, with τ_h to be the proper time for O_h. Then

$$\frac{d\tau_h}{dt} = (1 - \frac{r_s}{r_h})^{\frac{1}{2}}$$

As $r_h \to r_s$, $\frac{d\tau_h}{dt} \to 0$, *i.e.* compared to the time at $r = \infty$, the time at $r = r_h$ becomes infinitely slow. Consider some event of energy E_h happening at $r = r_h$, to O_∞ this event has energy

$$E_\infty = E_h f^{\frac{1}{2}}(r_h)$$

i.e. for fixed local proper energy E_h, $E_\infty \to 0$ as $r_h \to r_s$, we call it infinitely redshifted.

5. It takes a free-fall traveler a finite proper time to reach the horizon, but infinite Schwarzschild time.

6. Once inside the horizon, a traveler cannot send signals to outside, nor can she/he escape.

7. Two intrinsic geometric quantities of the horizon:

   - Area of a spatial section
   $$A = 4\pi r_s^2 = 16\pi G_N^2 M^2$$
   - Surface gravity
   The acceleration of a stationary observer at the horizon as measured by an observer at infinity is given by
   $$K = \frac{1}{2} f'(r_s) = \frac{1}{4G_N M}$$

   More details can be found in Ref. [1]

References

[1] Robert M. Wald, *General Relativity*, University Of Chicago Press (1984).

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