Last Time

RPI

$$p^\mu = \frac{n^\mu}{2}\,\bar{n}\cdot(p+k) + \frac{\bar{n}^\mu}{2}\,n\cdot k + (p_\perp^\mu + k_\perp^\mu)$$

- Any choice of basis vectors , $n^2 = 0 = \bar{n}^2$, $n\cdot\bar{n} = 2$
  equally good

I $\quad n \to n + \Delta_\perp$, $\quad \bar{n} \to \bar{n}$

II $\quad n \to n$, $\quad \bar{n} \to \bar{n} + \epsilon_\perp$

III $\quad n \to e^{\alpha} n$, $\quad \bar{n} \to e^{-\alpha}\bar{n}$

- Freedom in the component decomposition

$$\bar{n}\cdot(p+k)\,, \qquad p_\perp^\mu + k_\perp^\mu$$

$$\mathcal{P}_\mu \to \mathcal{P}_\mu + \beta_\mu \,, \quad i\partial_\mu \to i\partial_\mu - \beta_\mu \qquad n\cdot\beta = 0$$

$$\xi_{n,p}(x) \to e^{i\beta\cdot x}\,\xi_{n,p+\beta}(x)$$

Connects : $\quad \mathcal{P}^\mu + i\partial^\mu$

Gauge this $\quad \left.\begin{array}{l} iD_\perp^{c\mu} + W\, iD_\perp^{us\,\mu}\, W^\dagger \\ i\bar{n}\cdot D^c + W\, i\bar{n}\cdot D_{us}\, W^\dagger \end{array}\right\}$ nice properties under gauge symmetry

Modifies earlier attempt : - due to W's this is not $A_n^\mu + A_{us}^\mu$
- doesn't effect $n\cdot D$ in LO $\mathcal{L}$.

I, II, III leave $\quad V^\mu = \frac{n^\mu}{2}\,\bar{n}\cdot V + \frac{\bar{n}^\mu}{2}\,n\cdot V + V_\perp^\mu \quad$ invariant

III last time

Under I

$$n\cdot D \to n\cdot D + \Delta_\perp\cdot D_\perp$$
$$D^\perp_\mu \to D^\perp_\mu - \frac{\Delta^\perp_\mu}{2}\,\bar n\cdot D - \frac{\bar n_\mu}{2}\,\Delta^\perp\cdot D^\perp$$
$$\bar n\cdot D \to \bar n\cdot D$$
$$\xi_n \to \left(1 + \frac{\Delta\!\!\!/_\perp\,\bar n\!\!\!/}{4}\right)\xi_n$$
$$W \to W$$

Under II

$$n\cdot D \to n\cdot D$$
$$D^\perp_\mu \to D^\perp_\mu - \frac{\epsilon^\perp_\mu}{2}\,n\cdot D - \frac{n_\mu}{2}\,\epsilon^\perp\cdot D^\perp$$
$$\bar n\cdot D \to \bar n\cdot D + \epsilon_\perp\cdot D_\perp$$

$$\xi_n \to \left(1 + \frac{\epsilon\!\!/_\perp}{2}\,\frac{1}{i\bar n\cdot D}\,i D\!\!\!\!/_\perp\right)\xi_n$$

$$W \to \left[\left(1 - \frac{1}{i\bar n\cdot D}\,i\epsilon^\perp\cdot D_\perp\right)W\right]$$

Power Counting: max power that leaves scaling for collin momentum

$\epsilon_\perp \sim \lambda^0$, $\alpha \sim \lambda^0$

$\Delta_\perp \sim \lambda$ [else $n\cdot D \not\sim \lambda^2$]

eg.

$$\delta^{(I)}\left(\bar\xi_n\, i D\!\!\!\!/^c_\perp \frac{1}{i\bar n\cdot D}\, i D\!\!\!\!/^c_\perp \frac{\bar n\!\!\!/}{2}\,\xi_n\right) = -\bar\xi_n\, i\Delta^\perp\cdot D^\perp \frac{\bar n\!\!\!/}{2}\,\xi_n$$
$$\delta^{(I)}\left(\bar\xi_n\, i n\cdot D\, \frac{\bar n\!\!\!/}{2}\,\xi_n\right) = \underbrace{\bar\xi_n\, i\Delta^\perp\cdot D^\perp \frac{\bar n\!\!\!/}{2}\,\xi_n}_{\text{connected}}$$

type-II rules out $\bar\xi_n\, D_\perp^\mu \frac{1}{i\bar n\cdot D} D_\perp^\mu \frac{\bar n\!\!\!/}{2}\,\xi_n$ operator in $\mathcal{L}^{(0)}_{qq}$

So

$$\mathcal{L}^{(0)}_{qq} = \bar\xi_n\left[i n\cdot D + i D\!\!\!\!/^c_\perp \frac{1}{i\bar n\cdot D^c}\, i D\!\!\!\!/^c_\perp\right]\frac{\bar n\!\!\!/}{2}\,\xi_n$$

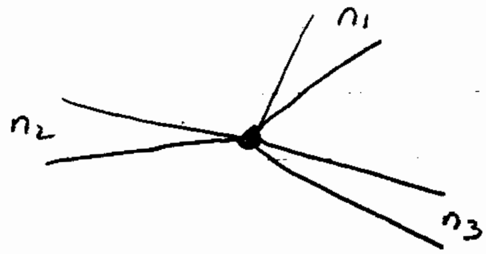
Unique by p.c., gauge inv, & RPI

More collinear fields: for >1 energetic hadron

or >1 " jet

need more n's ($\&\ \bar{n}$'s)

Generalize to $\sum_n \mathcal{L}^{(0)}_{\xi\xi}$



For $n_1, n_2, n_3, \ldots$ the modes are distinct only if

$n_i \cdot n_j \gg \lambda^2 \qquad i \neq j$

e.g. $\quad p_2 = Q\, n_2$

$n_1 \cdot p_2 = Q\, n \cdot n_2 \sim Q\lambda^2$ then p_2 is n_1-collinear

<u>Discrete Symmetries</u> $\qquad n = (1,0,0,1)\,,\ \bar{n} = (1,0,0,-1)$

$$C^{-1}\, \xi_{n,p}\, C = -\left[\bar{\xi}_{n,-p}\, \mathcal{C}\right]^T$$

$$P^{-1}\, \xi_{n,p}(x)\, P = \gamma_0\, \xi_{\bar{n},\tilde{p}}(x_P)$$

$$T^{-1}\, \xi_{n,p}(x)\, T = T\, \xi_{\bar{n},\tilde{p}}(x_T)$$

$p = (p^+, p^-, p_\perp)$

$\tilde{p} = (p^-, p^+, -p_\perp)$

$x_P = (x^-, x^+, -x_\perp)$

$x_T = (-x^-, -x^+, x_\perp)$

Study $\mathcal{L}_{qq}^{(0)}$

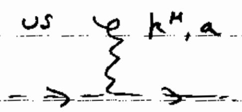
① Propagator

$$i\frac{\bar{\not{n}}}{2}\,\frac{\theta(\bar{n}\cdot p)}{n\cdot p+\frac{p_\perp^2}{\bar{n}\cdot p}+i\epsilon} + i\frac{\bar{\not{n}}}{2}\,\frac{\theta(-\bar{n}\cdot p)}{+n\cdot p+\frac{p_\perp^2}{\bar{n}\cdot p}-i\epsilon} = i\frac{\bar{\not{n}}}{2}\,\frac{\bar{n}\cdot p}{n\cdot p\,\bar{n}\cdot p+p_\perp^2+i\epsilon}$$

particles $\bar{n}\cdot p>0$ anti $\bar{n}\cdot p<0$ ✓ expn. of QCD

② Interactions

- only $n\cdot A_{us}$ gluons at LO



$= i g T^a n^\mu \frac{\bar{\not{n}}}{2}$

& only sees $n\cdot k$ usoft momentum (multipole expn.)

$$\frac{\bar{n}\cdot p}{\bar{n}\cdot p\; n\cdot(p+k)+p_\perp^2+i\epsilon} = \frac{\bar{n}\cdot p}{\bar{n}\cdot p\; n\cdot k+p^2+i\epsilon}$$

$$\underset{\text{on-shell}}{=} \frac{\bar{n}\cdot p}{\bar{n}\cdot p\; n\cdot k+i\epsilon}$$

(Compare Collinear Gluon  $\dfrac{\bar{n}\cdot(p+q)}{(p+q)^2+i\epsilon}$)

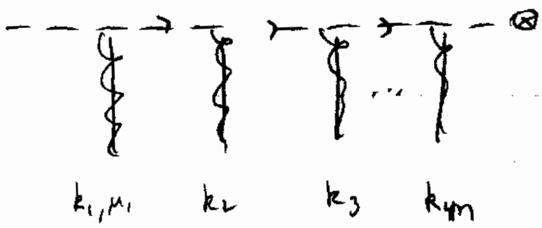
Propagator reduces to eikonal approx when appropriate

$\bar{n}\cdot p>0$ $\bar{n}\cdot p<0$

$\dfrac{n^\mu}{n\cdot k+i\epsilon}$ $\dfrac{n^\mu}{-n\cdot k+i\epsilon}$ $\dfrac{n^\mu}{-n\cdot k-i\epsilon}$ $\dfrac{n^\mu}{n\cdot k-i\epsilon}$

Usoft - Collinear Factorization

Consider



$$= \Gamma \sum_m \sum_{\text{perms}} \frac{(-g)^m \, n\cdot A^{a_1} \cdots n\cdot A^{a_m} T^{a_m} \cdots T^{a_1}}{n\cdot k_1 \; n\cdot(k_1+k_2) \cdots n\cdot(\Sigma k_i)} \times u_n$$

on-shell so $\dfrac{1}{n\cdot k + \frac{p_\perp^2}{\bar n\cdot p}} \to \dfrac{1}{n\cdot k}$

Motivates us to consider a field redefinition

$$\xi_{n,p}(x) = Y(x)\, \xi_{n,p}^{(0)}(x) \qquad A_{n,p} = Y A_{n,p}^{(0)} Y^\dagger$$

adjoint version

$$Y(x) = P \exp\left(ig \int_{-\infty}^{0} ds \; n\cdot A_{us}^{a}(x+ns)\, T^a \right)$$

$$n\cdot D\, Y = 0 \quad , \quad Y^\dagger Y = 1 \qquad \text{find} \quad W = Y W^{(0)} Y^\dagger$$

$$\begin{aligned}
\mathcal{L}_{\xi\xi}^{(0)} &= \bar\xi_{n,p'} \frac{\bar{n}\!\!\!/}{2} \left[in\cdot D + \cdots \right] \xi_{n,p} \\
&= \bar\xi_{n,p'}^{(0)} \frac{\bar{n}\!\!\!/}{2} \left[Y^\dagger in\cdot D_{us} Y + Y^\dagger (Y g\bar n\cdot A_n Y^\dagger) Y + \cdots \right] \xi_{n,p} \\
&= \bar\xi_{n,p'}^{(0)} \frac{\bar{n}\!\!\!/}{2} \Big[\underbrace{in\cdot\partial + g\bar n\cdot A_n}_{in\cdot D_c} + \cdots \Big] \xi_{n,p}
\end{aligned}$$

all $n\cdot A_{us}$'s disappear!

True for gluon action too

$$\mathcal{L}(\xi_{n,p}, A_{n,q}^{\mu}, n\cdot A_{us}) = \mathcal{L}(\xi_{n,p}^{(0)}, A_{n,q}^{(0)}, 0)$$

Interactions don't disappear, but are moved out of L.O. $\mathcal{L}$ and into currents eg 1 $\qquad J = \bar{\xi} W \Gamma h_v = \bar{\xi}_n^{(0)} Y^\dagger Y W^{(0)} Y^\dagger \Gamma h_v$

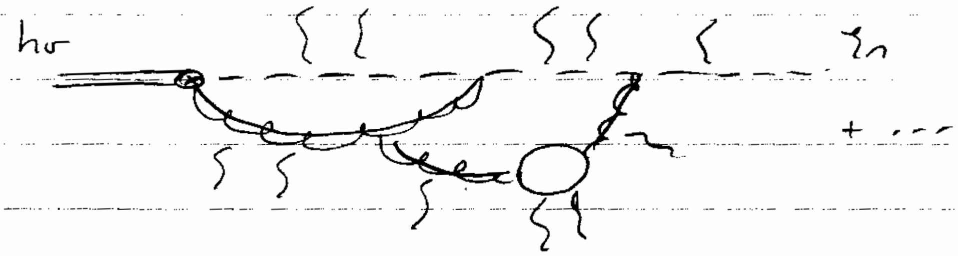
$$= (\bar{\xi}_n^{(0)} W^{(0)}) \Gamma (Y^\dagger h_v)$$

If our current was a collinear color singlet

eg 2 $\qquad J = (\bar{\xi}_n W) \Gamma (W^\dagger \xi_n) = \bar{\xi}_n^{(0)} W^{(0)} \cancel{Y^\dagger} \cancel{Y} \Gamma (W^{\dagger(0)} \xi_n^{(0)})$

Quite powerful, sums an ∞ class of diagrams

in eg 1



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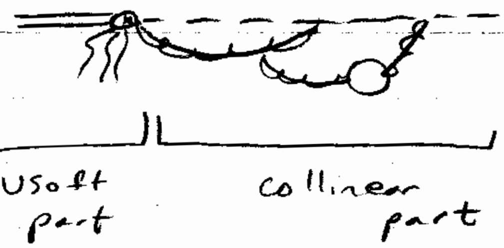


in eg 2 usoft gluons decouple at L.O. from any graph
This is color transparency

- usoft gluons decouple from energetic partons in color singlet state
- they just "see" overall color singlet due to multipole expansion

What about Wilson Coefficients?

have $C(\bar{P}, \mu)$ ie depend on large momenta picked out by label operator $\bar{P} \sim \lambda^0$

eg. $C(-\bar{P},\mu)\,(\bar{\xi}_n W)\,\Gamma h_v = (\bar{\xi}_n W)\,\Gamma h_v\, C(\bar{P}^+)$

<u>must</u> act on product $(\bar{\xi} W)$ since only momentum of this combination is gauge invariant

Write $(\bar{\xi} W)\,\Gamma h_v\, C(\bar{P}^+) = \int d\omega\, C(\omega,\mu)\,\left[(\bar{\xi} W)\,\delta(\omega - \bar{P}^+)\,\Gamma h_v\right]$

$$= \int d\omega\; C(\omega,\mu)\;\; O(\omega,\mu)$$

convolution (as promised)

<u>Hard-Collinear</u> <u>Factorization</u> of "C" and collinear "O"

Recall defn of W, $i\bar{n}\cdot D_c W = 0$, $W^\dagger W = 1$

as operator
$$i\bar{n}\cdot D_c\, W = W\,\bar{P}$$
$$i\bar{n}\cdot D_c = W\,\bar{P}\,W^\dagger$$
$$(i\bar{n}\cdot D_c)^k = W\,\bar{P}^k\,W^\dagger$$

$$f(i\bar{n}\cdot D_c) = W\, f(\bar{P})\, W^\dagger$$

trades $\bar{n}\cdot A \to W$

Part of collin op. $p^2 \sim \lambda^2 Q^2$

hard coefficient

$$= \int d\omega\;\; f(\omega)\;\; W\,\delta(\omega - \bar{P})\,W^\dagger$$

In general define $\quad \chi_n = (W^\dagger \xi_n)$

$$\chi_{n,\omega} = \delta(\omega - \bar{P})(W^\dagger \xi_n)$$

Operators $\quad \int d\omega_1\, d\omega_2 \quad \bar{\chi}_{n,\omega_1}\, \Gamma\, \chi_{n,\omega_2}$ $\quad$ etc.

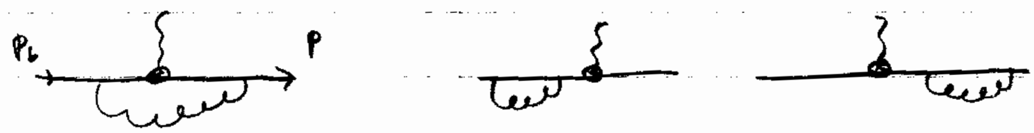
IR divergences, Matching, & Running

Consider heavy-to-light current for $b \to s\gamma$

$J^{QCD} = \bar{s}\,\Gamma\, b \qquad\qquad \Gamma = \sigma^{\mu\nu} P_R F_{\mu\nu}\ ,\ O_{7\gamma}$

$J^{SCET} = (\bar{\xi} W)\,\Gamma\, h_v\, C(\bar{P}^\dagger) \qquad$ (pre Y-field redefn)

QCD graphs at one-loop, take $p^2 \neq 0$ to regulate IR of collin-quark



$$= \quad -\bar{s}\,\Gamma\, b\ \frac{\alpha_s C_F}{4\pi}\left[\ln^2\!\left(\frac{-p^2}{m_b^2}\right) + 2\ln\!\left(\frac{-p^2}{m_b^2}\right) + \cdots\right]$$

$$Z_b = 1 - \frac{\alpha_s C_F}{4\pi}\left[\frac{1}{\epsilon_{UV}} + \frac{2}{\epsilon_{IR}} + 3\ln\frac{\mu^2}{m_b^2} + \cdots\right] \quad \leftarrow \text{IR reg. by D.R. here}$$

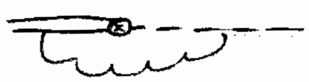
$$Z_s = 1 - \frac{\alpha_s C_F}{4\pi}\left[\frac{1}{\epsilon_{UV}} - \ln\frac{p^2}{\mu^2}\right]$$

full Z's, not $\overline{MS}$
match - S-matrix

$$Z_{tensor} = 1 + \frac{\alpha_s C_F}{4\pi}\,\frac{1}{\epsilon} \qquad \text{tensor current not conserved}$$

$$\text{Sum} = \bar{s}\,\Gamma\, b\left[1 - \frac{\alpha_s C_F}{4\pi}\left(\ln^2\!\left(\frac{-p^2}{m_b^2}\right) + \frac{3}{2}\ln\!\left(\frac{-p^2}{m_b^2}\right) + \frac{1}{\epsilon_{IR}} + \cdots\right.\right]$$

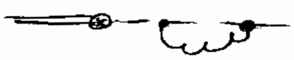
$\boxed{\text{usoft}}$



usoft

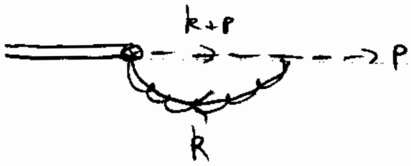
$$\int \frac{d^dk \qquad n\cdot v}{(v\cdot k + i\epsilon)\,(k^2+i\epsilon)\,(n\cdot k + p^2/\bar n\cdot p + i\epsilon)}$$

$$= -\bar\xi\,\Gamma\,h_v\;\frac{\alpha_s C_F}{4\pi}\left[\frac{1}{\epsilon^2} + \frac{2}{\epsilon}\ln\left(\frac{\mu\,\bar n\cdot p}{-p^2-i\epsilon}\right) + 2\ln^2\left(\frac{\mu\,\bar n\cdot p}{-p^2}\right) + \frac{3\pi^2}{4}\right]$$

 $\propto\; n^\mu n_\mu = 0$ Feyn. Gauge

 $Z_{HQET} = 1 + \frac{\alpha_s C_F}{4\pi}\left[\frac{2}{\epsilon_{UV}} - \frac{2}{\epsilon_{IR}}\right]$

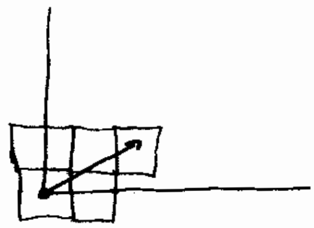
$\boxed{\text{Collinear}}$ Graphs



$$= \sum_{\substack{k\neq 0\\ k\neq -p}} \int d^dk_r \;\frac{n\cdot\bar n \qquad \bar n\cdot(p+k)}{\bar n\cdot k \quad k^2 \quad (k+p)^2}$$

each has label & residual
(k, k_r)

recall grid



if $k=0$, gluon is usoft

Grid is like Wilsonian EFT
To make it Continuum

$$\sum_{k\neq 0}\int d^dk_r \;F(k,p,k_r) = \int d^dk\left[F(k,p) - F^{subt}(k,p)\right]$$

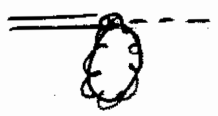
$k=-p$ usoft quark (harmless)

k scales towards usoft

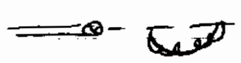
$$\frac{n\cdot\bar n \quad \bar n\cdot p}{\bar n\cdot k \;\; k^2\;(n\cdot k\,\bar n\cdot p + p^2)}$$

$\propto \frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}}$

$$= -\bar\xi\,\Gamma\,h_v\;\frac{\alpha_s C_F}{4\pi}\left[-\frac{2}{\epsilon^2} - \frac{2}{\epsilon} - \frac{2}{\epsilon}\ln\left(\frac{\mu^2}{-p^2}\right) - \ln^2\left(\frac{\mu^2}{-p^2}\right) - 2\ln\left(\mu^2/-p^2\right) - 4 + \pi^2/6\right]$$



$\propto n\cdot\bar{n} = 0$



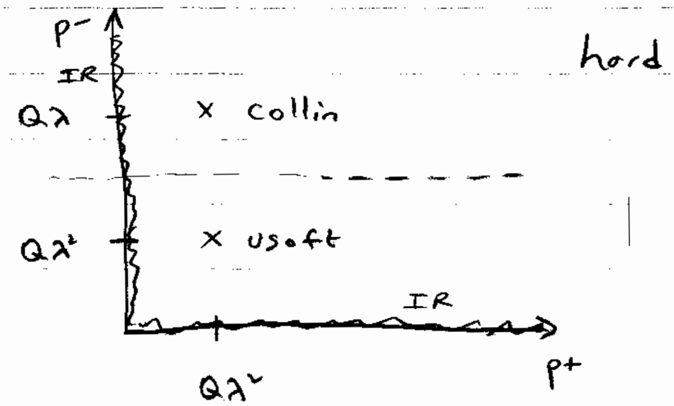
$$Z = 1 - \frac{\alpha_s C_F}{4\pi}\left[\frac{1}{\epsilon_{UV}} + \ln\frac{\mu^2}{p^2}\right]$$

IR matches $\ln^2(p^2)$ QCD = SCET

$\ln(p^2)$ "

$1/\epsilon_{IR}$ "

If we had neglected collinear graphs this would not be true [historically LEET ...]



degrees of freedom tile momentum space while maintaining p.c.

UV divergences in SCET need a c.t.

$$Z = 1 + \frac{\alpha_s C_F}{4\pi}\left[\frac{1}{\epsilon^2} + \frac{2}{\epsilon}\ln\left(\frac{\mu}{\bar{n}\cdot p}\right) + \frac{5}{2\epsilon}\right]$$

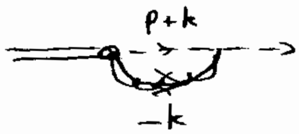
$\uparrow_{LL}$ $\uparrow_{\text{part of NLL}}$

<u>Running</u>

In general we must be careful with coeffs since they act like operators $C(\mu, \bar{P})$

In our eg. $\bar{P} \to \bar{n}\cdot p$ of external field always

non-trivial case



$$C(\mu, \bar{n}\cdot(p+k) + \bar{n}\cdot(-k)) = C(\mu, \bar{n}\cdot p)$$

$$\mu \, d/_{d\mu} \; C(\mu) = -\frac{\alpha_s(\mu)\, C_F}{\pi} \ln\left(\frac{\mu}{\bar{P}}\right) C(\mu)$$

LO anom dim

Soln. QED α_s = fixed, $C_F = 1$

$$C(\mu) = \exp\left[-\frac{\alpha}{2\pi} \ln^2\left(\frac{\mu}{\bar{P}}\right)\right]$$

Sudakov Suppression

QCD

$$C(\mu) = \exp\left[\frac{-4\pi C_F}{\beta_0^2\, \alpha_s(m_b)} \left(\frac{1}{z} - 1 + \ln z\right)\right]$$

$$z = \frac{\alpha_s(\mu)}{\alpha_s(m_b)}$$

here m_b = matching scale

In more complicated cases $C(\bar{P}, \bar{P}^+)$ will be sensitive to $\bar{n}\cdot k$ loop momentum and we'll get

$$\mu \frac{\partial}{\partial\mu} C(\mu, \omega) = \int d\omega' \; \gamma(\omega, \omega') \; C(\mu, \omega')$$

examples DIS Alterelli-Parisi evolution

$\gamma^* \pi^0 \to \pi^0$ Brodsky-Lepage "

$\gamma^* p \to \gamma p'$ Deeply-Virtual Compton Scattg

these are actually all the evolution of a single SCET operator

$$(\bar{\xi}_n W)\, C(\bar{P}, \bar{P}^\dagger)\, (W^\dagger \xi_n)$$

Note: series in $\ln C(\mu)$

		one-loop	two-loop	3-loop
LL	$\alpha_s^n \ln^{n+1}$	$1/\epsilon^2$	—	—
NLL	$\alpha_s^n \ln^n$	$1/\epsilon$	$1/\epsilon^2$	—
NNLL	$\alpha_s^n \ln^{n-1}$	matching	$1/\epsilon$	$1/\epsilon^2$

$1/\epsilon^2 \to 1/\epsilon \ln(\mu)$ term

Differs from single log case somewhat

At LHC, Sudakov effects are important in

Parton showers [Prob. to evolve without branching]

Jets